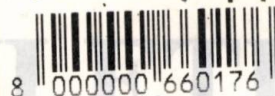


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**MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ**

1997

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1997

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ON SPECIAL TYPE OF RELATION IN GROUPS

BY

W.B. VASANTHA KANDASAMY

Abstract. New type of relations in groups and results about them are studied.

Key words: group, relation between group elements.

1. Introduction

In this Note we define a new type of relation in groups between its elements and obtain some interesting relations and results about them. For more about groups in general please refer [1].

Definition 1. Let G be a group. Let $a, b \in G$ and N a subset of G not containing a and b . We say a is right related to b with respect to the set N if and only if $a = b$ or $a \in Nb$ and $b \in Na$. Similarly we define left relation.

If a is both right and left related then we say a and b are related with respect to N .

Remark. (i) It is important and interesting to note all subsets of $G \setminus \{a, b\}$ need not contribute to the relation between a and b .

(ii) If G is commutative the left and right relation always coincide.

Example 1. Let $G = \langle g/g^4 = 1 \rangle$ and $g, g^3 \in G$. Take $N = \langle g^2 \rangle$. Clearly g and g^3 are related with respect to N . If $N' = \langle 1 \rangle$ then g and g^3 are not related with respect to N : If $M = \langle 1, g^2 \rangle$ then also g and g^3 are related. Thus the subset N need not always be unique.

Example 2. Let

$$S_3 = \left\{ e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \right.$$

$$\left. p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}.$$

Let $p_1, p_3 \in S_3$; take $N = \langle p_4 \rangle$ we see $p_1 \in Np_3$ and $p_3 \notin Np_1$ so p_1 and p_3 are one way related but are not related with respect to N . Now consider $M = \langle p_4, p_5 \rangle$, then p_1 and p_3 are M related.

Definition 2. Let G be a group and $a, b \in G$; N a subset of G not containing a and b . If $a \in Nb$ and $b \in Na$ then we say a is semi related to b with respect to N .

If $b \in Na$ and $a \notin Nb$ then we say b is semirelated to a with respect to N .

Proposition 3. Let G be a group $a, b \in G$ and N a subset of $G \setminus \{a, b\}$. If a and b are related with respect to N then a and b also b and a are semirelated with respect to N .

Proof. Obvious by the Definitions 1 and 2.

Corollary. If G be a group and $a, b \in G$ then if a and b are semi related with respect to a set N in $G \setminus \{a, b\}$; then a and b in general need not be related with respect to N .

Proof. By an example. Take the group S_3 in Example 2. Clearly for $p_1, p_3 \in S_3$ and $p_4 = N \subseteq S_3 \setminus \{p_1, p_3\}$ we have $p_1 \in Np_3$ but $p_3 \notin Np_1$. Hence the claim.

Theorem 4. Let G be a group; $a, b \in G$ and $N \subseteq G \setminus \{a, b\}$ a is related to b with respect to N then either there exists a $n \in N$ with $n^2 = 1$ or N has a pair of elements s, t with $st = 1$.

Proof. Clear from the fact; $a \in Nb$ and $b \in Na$ imply $a = tb$ and $b = sa$ so that $a = (ts)a$ or $ts = 1$ or if $s = t$ then $t^2 = 1$.

Remark. Even if N contains an element with $m^2 = 1$ still an arbitrary a and b may not in general be related at all. For take p_1, p_3 in the Example 2, if $N = \{p_2\}$ p_1 and p_3 are not related with respect to N .

Notation: Let G be a group $a, b \in G$. $N \subseteq G \setminus \{a, b\}$ we denote a related to b with respect to N by $a \sim_N b$ or $b \sim_N a$.

Theorem 5. Let G be a group. If $g \in G$ such that $g^2 = 1$ then the pair $(g, 1)$ can never be a related pair with respect to any subset $N \subseteq G \setminus \{1, g\}$.

Proof. Clearly from the fact for any $N \subseteq G \setminus \{1, g\}$ $1 \neq ng$ for if $1 = ng \Rightarrow g = ng^2$ ie. $g = n$ but $g \notin N$ a contradiction.

Theorem 6. Let G be a group and $a, b \in G$. For $N \subseteq G \setminus \{a, b\}$ a and b be right (or left) related with respect to N if and only if a^{-1} and b^{-1} is left (or right) related with respect to N .

Proof. Clear from the fact $a = nb$ and $nn'a$ imply $b^{-1} = a^{-1}n$ and $a^{-1} = b^{-1}n'$; given $a, b \in G$ are related with respect to N .

Example 3. Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ be the group of real quaternions. Clearly $i, j \in G$ are related with respect to $N = \{\pm k\}$, likewise $j, k \in G$ are related with respect to $M = \{\pm i\}$ and $i, k \in G$ are related with respect to $P = \{\pm j\}$. So $i \sim_N j$, $j \sim_M k$ and $i \sim_P k$ with N, M and P distinct subsets of G .

Now the following example gives another extreme case.

Example 4. Take S_3 to be the group given in Example 2. Clearly $p_1, p_2 \in S_3$ are related with $N = \{p_4, p_5\}$, $p_2, p_3 \in S_3$ are related with $N = \{p_4, p_5\}$ and $p_1, p_3 \in S_3$ are related with $N = \{p_4, p_5\}$. Thus $p_1 \sim_N p_2, p_2 \sim_N p_3$ and $p_1 \sim_N p_3$. Now observing for every $a \in G \setminus \{1\}$ $a \sim_N a$ for take $N = \{1\}$, if $a \sim_N b$ then $b \sim_N a$. But even if $a \sim_N b$ and $b \sim_M c$ need not always imply $a \sim c$ and $M \subseteq N$ or $N \subseteq M$, from the above examples. Hence we prove the following

Theorem 7. Let G be a group and $a, b \in G$ with $a \sim_N b$ and $b, c \in G$ with $b \sim_M c$. That is $a = nb$ and $b = n'a$; $b = mc$ and $c = m'b$. Then $m'n'$ and $nm \subseteq N$ or M and $N \subseteq M$ or $M \subseteq N$ or $N = M$.

Proof. Now $a \sim_N b$ and $b \sim_M c$; so that $a = nb$ and $b = n'a$ for $n, n' \in N$ and $b = mc$ and $c = m'b$ for $m, m' \in M$. Hence $c = m'b = m'(n'a) = m'n'a$ i.e., $c = m'n'a$. (Clearly $m'n' \neq b$ $a = nb = n(mc) = nmc$ with $nm \neq b$). (For if nm or $m'n' = b$ then $b \in N$ or M).

Now if $m'n'$ and $nm \in M$ or N then we have transitivity; other-wise not. So transitivity in general is not true in this relation.

Example 5. Consider S_3 given in Example 2. Clearly p_4 and p_5 are never N related for any $N \subseteq S_3 \setminus \{p_4, p_5\}$. It is easy verified even if $N = \{1, p_2, p_3, p_1\}$ then also p_4 never related to p_5 for any N .

Problem. Do we have any method to find out whether a pair of elements is related or not?

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REFERENCES

1. Hall M., *The Theory of Groups*. Macmillan, 1959.

ASUPRA UNOR TIPURI SPECIALE DE RELAȚII ÎN GRUPURI

(Rezumat)

Se definesc noi tipuri de relații între elementele grupurilor și se stabilesc o serie de proprietăți și rezultate corespunzătoare.

SYSTEMS OF INDEPENDENCE ASSOCIATED TO A GRAPH

BY

EMILIA POPOVICI

Abstract. Two systems of independence associated to a graph are defined and a new interpretation of the χ -free restriction theorem is given.

Key words: hereditary properties; χ_P - perfection.

1. Introduction

A generalization of the perfect graphs was defined by Ann T r e n k B a r r e t t [1], a generalization that she called χ_P -perfection. At the same time she characterizes the set of χ_P -perfect graphs, noted by P^* , in terms of forbidden induced subgraphs, for a number of hereditary properties P , characterizations which are analogous to Strong Perfect Graph Conjecture (SPGC). She shows that it suffices to find the classes $(\text{Free}(K_n))^*$ for any $n \geq 2$, which, together with χ -Free Restriction Theorem, can characterize P^* for every hereditary property P .

L. C a i and D. C o r n e i l [2] are also studying a generalization of perfection which they call i -perfection. The notion of " i -perfection" is the same as " χ_P -perfection" when we restrict attention to the properties $P = \text{Free}(K_{i+1})$.

In this paper there are defined two systems of independence associated to a graph, for a hereditary property P , and there is given a new interpretation of the χ -Free Restriction Theorem and of L. C a i and D. C o r n e i l's theorem, using these systems of independence.

2. Notations and Definitions

All graphs in this paper are assumed to be finite and undirected, without loops, without multiple edges and with nonempty vertex sets.

If H is an induced subgraph of $G = (V, E)$ then we write $H \leq G$. A graph G is perfect if $\chi(H) = \omega(H)$, $(\forall) H \leq G$.

When we refer to a property P of the graphs, we actually refer to a class or to a

family of graphs which is defined as being a set of graphs which have the property P . A nontrivial property is one which does not coincide with the empty set.

We say that P is a *hereditary property* if it is closed at the set of induced subgraphs (i.e., if $G \in P$ and $H \leq G$ together imply that $H \in P$). All properties we consider will be hereditary and nontrivial.

If P is a hereditary property, G is called a *minimal forbidden graph* for P (or just forbidden) if $G \notin P$ but $H \in P$ for all proper induced subgraphs, $H < G$. Denote the set of minimal forbidden graphs for P by $\text{Forb}(P)$. Thus

$$\text{Forb}(P) = \{G \mid G \notin P \text{ but } H \in P; (\forall) H < G\}.$$

We denote with $\text{Free}(\mathcal{F})$ a family of graphs with the property that, none of the graphs in this family contains a induced subgraph isomorphically with one graph of the family $\mathcal{F}$, that is

$$\text{Free}(\mathcal{F}) = \{G \mid F \not\leq G, (\forall) F \in \mathcal{F}\}.$$

The P -chromatic number of a graph G , denoted χ_P , is the minimum size of a partition $V(G) = V_1 \cup V_2 \cup \dots \cup V_k$ so that for each i we have $G[V_i] \in P$. The partition itself is called a P -coloring using k colors. When $P = \{\text{edgeless graphs}\}$, $\chi_P(G)$ is reduced to the chromatic ordinary number χ .

The P -clique number of a graph G , denoted $\omega_P(G)$, is defined as

$$\omega_P(G) = \max \{\chi_P(K) \mid K \leq G, K \text{ is a clique}\}.$$

When $P = \{\text{edgeless graphs}\}$, $\omega_P(G)$ is reduced to ω .

A graph G is χ_P -perfect if $\chi_P(H) = \omega_P(H)$, $(\forall) H \leq G$. This definition is a generalization of perfection, a generalization which is called χ_P -perfection. The set of χ_P -perfect graphs is noted by P^* . If a graph G is not in P^* , then we say that G is χ_P -imperfect. Moreover, if $G \notin P^*$, but $H \in P^*$, $(\forall) H < G$, then G is a *minimal χ_P -imperfect graph*.

$\chi_P(G)$ and $\omega_P(G)$ are well-defined for all hereditary properties P . Furthermore, if G is any graph, then $\chi_P(G)$ and $\omega_P(G)$ are integers between 1 and $|V(G)|$. For any hereditary property P , $\omega(P)$ is defined as the number of vertices in a largest clique in hereditary family P , if such a clique exists, and $\omega(P) = \infty$ otherwise.

3. χ -Free Restrictions and the Importance of the Classes K_n -Free in Characterizing the χ_P -Perfect Graphs

In this paragraph, there are given a few results which can be found in [1] and [2].

Theorem 1 (χ -Free Restriction). *Let P be a nontrivial hereditary property, and let χ be a set of graphs that satisfies following conditions:*

- 1) $\chi \subseteq P$;

2) $K_m \notin \chi$ for all m .

If $Q = P \cap \text{Free}(\chi)$ then $Q^* = P^* \cap \text{Free}(\chi)$, that is G is χ_Q -perfect iff G is χ_P -perfect and $G \in \text{Free}(\chi)$.

Corollary 1. If P and Q are nontrivial hereditary properties with $\omega(P) \leq \omega(Q)$ then

$$(P \cap Q)^* = P^* \cap \text{Free}(\chi) \quad \text{where} \quad \chi = \{G \mid G \in P \text{ and } G \notin Q\}.$$

Theorem 2. Let B be the set of properties $\{\text{Free}(K_n); n \geq 2\} \cup A$ where $A = \{\text{all graphs}\}$. Then all hereditary properties Q are either in B or can be written $Q = P \cap \text{Free}(\chi)$, where $P \in B$ and χ satisfies conditions 1) and 2) of Theorem 1 (of χ -Free Restriction). Moreover, none of the properties in B can be expressed as an χ -Free Restriction of any other property.

Out of this theorem results that any hereditary property Q may be written as it follows:

$$Q = A \cap \text{Free}(\chi), \quad \text{with} \quad \chi = \{G \mid G \notin Q\} \quad \text{or} \quad Q = (\text{Free}(K_n)) \cap \text{Free}(\chi),$$

where χ satisfies conditions 1) and 2) of the Theorem 1. According to the χ -Free Restriction Theorem, we have

$$Q^* = A^* \cap \text{Free}(\chi) \quad \text{or} \quad Q^* = (\text{Free}(K_n))^* \cap \text{Free}(\chi).$$

But $A = A^*$ because A contains all the cliques and then for first case we have $Q^* = Q$ for all χ -Free Restriction of A .

In the second case, there remains problem of the determinations of the family $(\text{Free}(K_n))^*$, for $n \geq 2$.

So, in order to characterize P^* for every hereditary property it suffices to find $(\text{Free}(K_n))^*$ for $n \geq 2$. A. Trenk Barrett obtained a partial characterization of these classes for $n \geq 3$ in [1]. Moreover, the characterization of the $(\text{Free}(K_n))^*$ class, which is a difficult problem, would resolve the SPGC problem.

Theorem 3 (L. Cai and D. Corneil). For all integers $n, m \geq 2$,

$$(\text{Free}(K_n))^* \subseteq (\text{Free}(K_m))^* \quad \text{iff} \quad (n-1) \mid (m-1).$$

Moreover, if $(n-1) \nmid (m-1)$ but $m \neq n$, then the inclusion $(\text{Free}(K_n))^* \subset (\text{Free}(K_m))^*$ is strict.

4. Systems of Independence Associated to the χ_P -Perfect Graphs

Let $IS = (E, \mathcal{F})$ be a system of independence and $X \subseteq E$. The restriction of the system of independence IS to X is the system of independence denoted by

$$IS/X = (X, \mathcal{F}_X), \quad \text{where} \quad \mathcal{F}_X = \{F \mid F \subseteq X, F \in \mathcal{F}\}.$$

If $IS = (E, \mathcal{F})$ is a system of independence and $\chi \subseteq \mathcal{F}$, then we define the restriction of the system of independence IS to $\text{Free}(\chi)$ in the following way:

$$IS/\text{Free}(\chi) = (E, \mathcal{F}_{\text{Free}(\chi)}), \quad \text{where } \mathcal{F}_{\text{Free}(\chi)} = \mathcal{F} \cap \text{Free}(\chi).$$

$IS/\text{Free}(\chi)$ is indeed a system of independence.

As $F_1 \subseteq F$ there results that also $F_1 \in \mathcal{F}$ and $F_1 \in \text{Free}(\chi)$. So that $F_1 \in \mathcal{F} \cap \text{Free}(\chi)$. Thus $F_1 \in \mathcal{F}_{\text{Free}(\chi)}$.

If P is a nontrivial hereditary property of the graphs and G is a simple graph with the set of vertices V , we attach to G two systems of independence, denoted by IS and IS^* defined as:

$$IS = (V, \mathcal{F}_P) \quad \text{with } \mathcal{F}_P = \{X \mid X \subseteq V, X \leq G, \chi_P(X) = 1\},$$

$$IS^* = (V, \mathcal{F}_P^*) \quad \text{with } \mathcal{F}_P^* = \{X \mid X \subseteq V, X \leq G, \chi_P(X) = \omega_P(X)\}.$$

We state some properties of these systems of independence:

1. The basis of IS are the maximal sets of the $B \subseteq V$ vertices which induce the subgraphs with the property $\chi_P(B) = 1$, and the circuits are the minimal sets of the $C \subseteq V$ vertices which induce the subgraphs with the property $\chi_P(C) > 1$.

2. For IS^* , the family of basis is made of maximal sets of $B^* \subseteq V$, and the family of circuits is made of the minimal sets of $C^* \subseteq V$ vertices which induce minimal subgraphs χ_P -imperfect.

3. $G \in \mathcal{F}_P$ iff $\chi_P(G) = 1$, that is G has the property P iff V is the only base of IS .

4. $G \in \mathcal{F}_P^*$ iff $\chi_P(G) = \omega_P(G)$, that is G is χ_P -perfect iff V is the only base of IS^* .

5. Any independent set of IS is an independent set of IS^* . Indeed, let $X \in \mathcal{F}_P$ be any independent set of IS . Thus $\chi_P(X) = 1$ with $X \leq G$. But $\omega_P(X) = \max\{\chi_P(K) \mid K \leq X \text{ and } K \text{ is a clique}\}$ and then $\omega_P(X) = 1$. Thus $\chi_P(X) = \omega_P(X)$ and $X \in \mathcal{F}_P^*$.

6. For $P = \{\text{Free}(K_2)\}$ hereditary property, the two systems of independence attached to G become:

i) $IS_2 = (V, \mathcal{S})$ where $\mathcal{S}$ is the family of the stable sets from G and, respectively,

ii) $IS_2^* = (V, \mathcal{S}^*)$ where $\mathcal{S}^*$ is the family of subsets from V which induce perfect subgraphs, or $\mathcal{S}^* = (\text{Free}(K_2))^*$, notation used in [1]. Thus $G \in \mathcal{S}^*$ iff $\omega(G) = \chi(G)$.

The following theorem is an interpretation of theorem described in the third paragraph and it is proved using the two systems of independence associated to a graph.

Theorem 4. Let P be a hereditary property of the graphs and IS, IS^* the two systems of independence associated to a simple graph $G = (V, E)$. If $\chi \subseteq \mathcal{F}_P$ and χ does not contain a clique then the following relation takes place:

$$IS^*/\text{Free}(\chi) = (IS/\text{Free}(\chi))^*.$$

Proof. We have $IS = (V, \mathcal{F}_P)$ and $IS^* = (V, \mathcal{F}_P^*)$. Let χ be a family of graphs that satisfies conditions from the hypothesis. According the definition of the restriction of a independence system at $\text{Free}(\chi)$, we have:

$$IS/\text{Free}(\chi) = (V, \mathcal{F}_P \cap \text{Free}(\chi)), \quad \text{and} \quad (IS/\text{Free}(\chi))^* = (V, (\mathcal{F}_P \cap \text{Free}(\chi))^*).$$

Considering the hypothesis, the conditions 1) and 2) of the Theorem 1 of χ -free restriction are satisfied and, according to this theorem, we have $(\mathcal{F}_P \cap \text{Free}(\chi))^* = \mathcal{F}_P^* \cap \text{Free}(\chi)$. Then,

$$(IS/\text{Free}(\chi))^* = (V, \mathcal{F}_P^* \cap \text{Free}(\chi)) = IS^*/\text{Free}(\chi).$$

The family $(\text{Free}(K_n))^*$ of the graphs, named χ_n -perfection, has a special importance in the characterisation of the χ_P -perfect graphs for any property P , as it can be seen in [1].

We note with IS_n and IS_n^* the two systems of independence attached to a graph when $P = \{\text{Free}(K_n), n \geq 2\}$.

Theorem 5. For any integer numbers $n, m \geq 2$, any independent set of IS_n^* is an independent set of IS_m^* iff $(n-1)|(m-1)$ and $n \neq m$.

Proof. We have $IS_n^* = (V, (\text{Free}(K_n))^*)$ and $IS_m^* = (V, (\text{Free}(K_m))^*)$. Let F be an independent set of IS_n^* , $F \in (\text{Free}(K_n))^*$. According to L. Cai and Corneil's theorem $(\text{Free}(K_n))^* \subset (\text{Free}(K_m))^*$ iff $(n-1)|(m-1)$, $m \neq n$. Then $F \in (\text{Free}(K_m))^*$ iff $(n-1)|(m-1)$, that is $F \in IS_m^*$ iff $(n-1)|(m-1)$, $m \neq n$.

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REFERENCES

1. Trenk Barrett A., *Generalized Perfect Graphs: Characterizations*. Preprint, Department of Mathematics and Computer Science, Dartmouth College Hanover, U.S.A., 1991.
2. Cai L., Corneil D., *On i -Perfect Graphs*. Preprint, 1990.
3. Tucker A., *Critical Perfect Graphs and Perfect 3-Chromatic Graphs*. J. Combin. Theory (B), **23**, 143-149 (1977).
4. Brown J., Corneil D., *On Generalized Graph Colorings*. Journal of Graph Theory, **11**, 87-89 (1987).
5. Lovasz L., *A Characterization of Perfect Graphs*. J. Combin. Theory (B), **13**, 95-98 (1972).

SISTEME DE INDEPENDENȚĂ ASOCIATE UNUI GRAF

(Rezumat)

Se definesc două sisteme de independență asociate unui graf. Pe baza acestora se dă o nouă interpretare proprietăților și caracterizărilor grafurilor χ_P -perfecte.

LES TYPES DE CIRCULATIONS ADMISSIBLES DANS UN GHAPHE ESCALIER NON ORIENTÉ (V)

PAR

VIOREL GHIUȘ

Abstract. The paper studies an admissible circulation of type $\overline{c\bar{b}'/b'c}$ [2] from a non-oriented staircase graph and proves a necessary and sufficient condition for existence.

Key words: graph, non-oriented staircase graph, admissible circulation.

Soit un graphe escalier [4] (Fig. 1) où les arêtes sont non orientées; les flèches du graphe indiquent le sens de la circulation qui sera introduite plus tard.

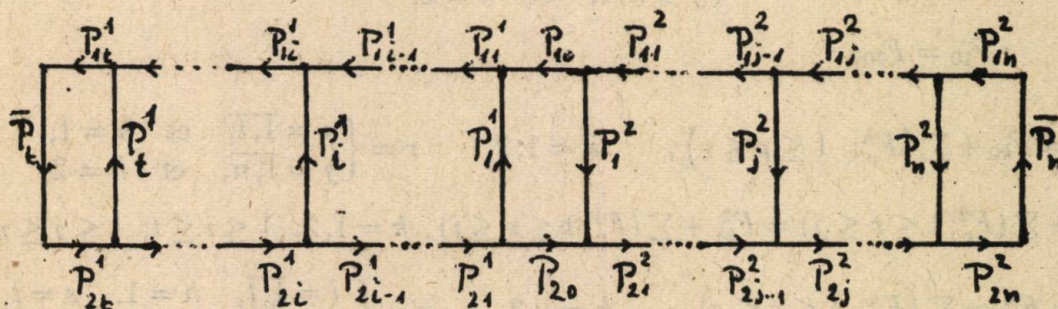


Fig. 1 — G^f : avec $f = \overline{c\bar{b}'/b'c}$.

Nous noterons les fonctions de capacité et de circulation comme suit:

$$c(P_r^h) = c_r^h, \quad C(P_r^h) = C_r^h, \quad F(P_r^h) = F_r^h, \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2; \end{cases}$$

$$c(P_{kr}^h) = c_{kr}^h, \quad C(P_{kr}^h) = C_{kr}^h, \quad F(P_{kr}^h) = F_{kr}^h, \quad k = 1, 2, \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2; \end{cases}$$

$$c(\overline{P}_s) = \overline{c}_s, \quad C(\overline{P}_s) = \overline{C}_s, \quad F(\overline{P}_s) = \overline{F}_s, \quad s = t, n;$$

$$c(P_{k0}) = c_{k0}, \quad C(P_{k0}) = C_{k0}, \quad F(P_{k0}) = F_{k0}, \quad k = 1, 2.$$

Définition 1. On appelle une circulation dans un graphe escalier G , une paire (f, F) , où f est une orientation $\overline{c\bar{b}'/b'c}$ ou $\overline{c'b/b'c}$ (équivalentes) [2] de G

(Fig. 1) et F est une circulation dans G^f (le graphe orienté attaché), c'est à dire sont satisfaites les équations de conservation:

$$(1) \quad \begin{cases} \bar{F}_s = F_{1s}^h = F_{2s}^h, & s = \begin{cases} t & \text{et } h = 1, \\ n & \text{et } h = 2; \end{cases} \\ F_{kr}^h = F_{kr-1}^h + F_r^h, & k = 1, 2, \quad r = \begin{cases} i = \overline{2, t}, & \text{et } h = 1, \\ j = \overline{2, n}, & \text{et } h = 2; \end{cases} \\ F_{k1}^h = F_1^h + F_{k0}, & k = 1, 2 \quad \text{et } h = 1, 2. \end{cases}$$

Cette circulation est admissible si elle satisfait les conditions:

$$(2) \quad \begin{cases} c_{kr}^h \leq F_{kr}^h \leq C_{kr}^h, & k = 1, 2, \quad r = \begin{cases} i = \overline{1, t}, & \text{et } h = 1, \\ j = \overline{1, n}, & \text{et } h = 2; \end{cases} \\ c_r^h \leq F_r^h \leq C_r^h, & r = \begin{cases} i = \overline{1, t}, & \text{et } h = 1, \\ j = \overline{1, n}, & \text{et } h = 2; \end{cases} \\ \bar{c}_s \leq \bar{F}_s \leq \bar{C}_s, & s = t, n \quad \text{et } c_{k0} \leq F_{k0} \leq C_{k0}, \quad k = 1, 2. \end{cases}$$

Remarque 1. De (1), on vérifie par induction que

$$(3) \quad \begin{aligned} F_{1r}^h &= F_{2r}^h, \quad r = \begin{cases} i = \overline{t, 1}, & \text{et } h = 1, \\ j = \overline{n, 1}, & \text{et } h = 2; \end{cases} \\ F_{10} &= F_{20}, \\ F_{kr}^h &= F_{k0} + \sum (F_p^h; 1 \leq p \leq r), \quad k = 1, 2, \quad r = \begin{cases} i = \overline{1, t}, & \text{et } h = 1, \\ j = \overline{1, n}, & \text{et } h = 2; \end{cases} \\ F_{ki}^1 + \sum (F_r^2; 1 \leq r \leq j) &= F_{kj}^2 + \sum (F_r^1; 1 \leq r \leq i), \quad k = 1, 2, \quad 1 \leq i \leq t; 1 \leq j \leq n; \\ \bar{F}_s &= F_{ks}^h = \sum (F_r^h; 1 \leq r \leq s), \quad k = 1, 2, \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \quad s = t, \\ j = \overline{1, n}, & h = 2, \quad s = n; \end{cases} \end{aligned}$$

On définit les quantités:

$$\begin{aligned} c'_{h0} &= \max_k \{c_{k0}\}, \quad C'_{h0} = \min_k \{C_{k0}\}, \quad k = 1, 2, \quad h = 1, 2; \\ c'_{hr} &= \max \left(\max_k \{c_{kr}^h\}, \quad c_r^h + c'_{hr-1} \right) \\ C'_{hr} &= \min \left(\min_k \{C_{kr}^h\}, \quad C_r^h + C'_{hr-1} \right) \end{aligned} \quad \left. \vphantom{\begin{aligned} c'_{hr} \\ C'_{hr} \end{aligned}} \right\}, \quad k = 1, 2, \quad r = \begin{cases} i = \overline{1, t-1}, & h = 1, \\ j = \overline{1, n-1}, & h = 2; \end{cases}$$

$$\begin{aligned} c'_{hs} &= \max \left(\max_k \{c_{ks}^h\}, \quad c_r^h + c'_{hr}, \quad \bar{c}_s \right) \\ C'_{hs} &= \min \left(\min_k \{C_{ks}^h\}, \quad C_r^h + C'_{hr}, \quad \bar{C}_s \right) \end{aligned} \quad \left. \vphantom{\begin{aligned} c'_{hs} \\ C'_{hs} \end{aligned}} \right\}, \quad k = 1, 2, \quad s = \begin{cases} t, & h = 1, \quad r = t-1, \\ n, & h = 2; \quad r = n-1. \end{cases}$$

Remarque 2. Nous avons:

$$\left. \begin{aligned} c'_{hr-1} &\leq c'_{hr}, & r &= \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2; \end{cases} \\ c'_{hr} &\geq c'_{h0} + \sum (c_p^h; 1 \leq p \leq r) \\ C'_{hr} &\leq C'_{h0} + \sum (C_p^h; 1 \leq p \leq r) \end{aligned} \right\}, & r &= \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2; \end{cases} \\ c'_{hs} &\geq \sum (c_r^h; 1 \leq r \leq s) \\ C'_{hs} &\leq \sum (C_r^h; 1 \leq r \leq s) \end{aligned} \right\}, & r &= \begin{cases} i = \overline{1, t}, & h = 1, & s = t, \\ j = \overline{1, n}, & h = 2 & s = n. \end{cases}$$

Remarque 3. Généralement les conditions

$$(4) \quad c'_{hr} \leq C'_{hr}, \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2; \end{cases}$$

ne sont pas satisfaites.

T h é o r è m e. Une condition nécessaire et suffisante que dans un graphe escalier G^f (Fig. 1) existe une circulation (f, F) , où f est une orientation $c \ b' / b' c$ où $c' b / b' c'$ et F une circulation de la forme (1) satisfaisant (2), est que la condition (4) soit satisfaite.

N é c e s s i t é. De la Remarque 1 et les équations de conservation (1) nous avons

$$\left. \begin{aligned} F_{10} &= F_{20} \\ c_{k0} &\leq F_{k0} \leq C_{k0} \\ c_{k1}^h &\leq F_{k1} \leq C_{k1}^h \\ F_{k1}^h &= F_{k0} + F_1^h \end{aligned} \right\}, \quad k = 1, 2 \quad \text{et} \quad h = 1, 2 \Rightarrow \begin{cases} c'_{h0} \leq C'_{h0}, \\ c'_{h1} \leq C'_{h1}, \end{cases} \quad \text{par } h = 1, 2.$$

Nous avons

$$\left. \begin{aligned} c_r^h &\leq F_r^h \leq C_r^h \\ C_{kr}^h &\leq F_{kr}^h \leq C_{kr}^h \\ F_{kr}^h &= F_{kr-1}^h + F_r^h \end{aligned} \right\}, \quad k = 1, 2, \quad r = \begin{cases} i, & h = 1, \\ j, & h = 2, \end{cases}$$

et supposant

$$c'_{kr-1} \leq F_{kr-1}^h \leq C'_{kr-1}$$

il résulte (4) et finalement

$$\left. \begin{aligned} \overline{F}_s &= F_{1s}^h = F_{2s}^h \\ c'_{ks} &\leq F_{ks}^h \leq C'_{ks} \\ \overline{c}_s &\leq \overline{F}_s \leq \overline{C}_s \end{aligned} \right\}, \quad k = 1, 2, \quad s = \begin{cases} t, & h = 1, \\ n, & h = 2, \end{cases} \Rightarrow c'_{ks} \leq C'_{ks}, \quad k = \begin{cases} 1, & s = t, \\ 2, & s = n. \end{cases}$$

Suffisance. On définit la circulation admissible

$$\overline{F}_s = F_{ks}^h = a_s^h, \quad k = 1, 2, \quad s = \begin{cases} t, & h = 1, \\ n, & h = 2, \end{cases}$$

avec

$$c'_{hs} \leq a_s^h \leq C'_{hs}, \quad s = \begin{cases} t, & h = 1, \\ n, & h = 2. \end{cases}$$

Par induction, si

$$F_{kr}^h = a_r^h, \quad k = 1, 2, \quad r = \begin{cases} i < t, & h = 1, \\ j < n, & h = 2, \end{cases}$$

où

$$\max \left(\max_k \{c_{kr}^h\}, c_r^h + c'_{hr-1} \right) = c'_{hr} \leq a_r^h \leq C'_{hr} = \min \left(\min_k \{C_{kr}^h\}, C_r^h + C'_{hr-1} \right),$$

on définit

$$F_{kr-1}^h = a_{kr-1}^h, \quad k = 1, 2, \quad r = \begin{cases} i \geq 1, & h = 1, \\ j \geq 1, & h = 2, \end{cases}$$

avec

$$(5) \quad \max (c'_{hr-1}, a_r^h - C_r^h) \leq a_{r-1}^h \leq \min (C'_{hr-1}, a_r^h - c_r^h).$$

On vérifie qu'existe (5); en effet de $c_r^h + c'_{hr-1} \leq a_r^h \leq C_r^h + C'_{hr-1} \Rightarrow$

$$(6) \quad c'_{hr-1} \leq a_r^h - c_r^h,$$

$$(7) \quad a_r^h - C_r^h \leq C'_{hr-1};$$

de (6) et $c'_{hr-1} \leq C'_{hr-1}$ il résulte

$$(8) \quad c'_{hr-1} \leq \min (C'_{hr-1}, a_r^h - c_r^h).$$

De (7) et $a_r^h - C_r^h \leq a_r^h - c_r^h$ on obtient

$$(9) \quad a_r^h - C_r^h \leq \min (C'_{hr-1}, a_r^h - c_r^h).$$

De (8) et (9) il résulte (5).

On définit

$$F_r^h = a_r^h - a_{r-1}^h, \quad r = \begin{cases} i = \overline{t, 1}, & h = 1, \\ j = \overline{n, 1}, & h = 2, \end{cases}$$

Les équations de conservation (1) sont évidemment satisfaites. Les conditions admissibles (2) sont satisfaites parce que de (5) nous avons

$$a_r^h - C_r^h \leq a_{r-1}^h \leq a_r^h - c_r^h, \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2, \end{cases}$$

D'ici il résulte $c_r^h \leq a_r^h - a_{r-1}^h = F_r^h \leq C_r^h$.

Nous avons

$$c'_{hr} \leq a_r^h \leq C'_{hr} \Rightarrow c_{kr}^h \leq F_{kr}^h \leq C_{kr}^h, \quad k = 1, 2, \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2. \end{cases}$$

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BIBLIOGRAPHIE

1. F o r d L. R., F u l k e r s o n D. R., *Flots dans les graphes*. Gauthier-Vilars, 1967.
2. G h i u ș V., *O clasificare a circulațiilor din grafuri latici unidimensionale*. Lucrările sesiunii jubiliare „Gh. Asachi”, Iași, 10-12 septembrie, 1988.
3. G h i u ș V., *La circulation standard admissible en un graphe latticiel avec une dimension* (en cours d'apparition).
4. G h i u ș V., *Tipuri de circulații admisibile într-un graf latice unidimensional neorientat* (en cours d'apparition).

TIPURI DE CIRCULAȚII ADMISIBILE ÎNTR-UN GRAF SCARĂ NEORIENTAT (V)

(Rezumat)

Se studiază circulația admisibilă de forma $c \overline{b'}/\overline{b'c}$ [2] dintr-un graf scară neorientat și se găsește o condiție necesară și suficientă de existență.

INTEGRATING OSCILLATORY FUNCTIONS ON SYSTOLIC ARRAYS

BY

OCTAV BRUDARU

Abstract. The paper presents a systolic implementation of Fillon's formula for integrating oscillatory functions. The proposed systolic architecture has one clock tick as period.

Key words: numerical integration, systolic arrays.

1. Introduction

Integrals involving oscillatory functions often require special treatment. An appropriate method is Fillon's formula Scheid [1], p. 120). The aim of this part is to give a systolic design to compute the integrals

$$\int_{a(r)}^{b(r)} f_r(x) \sin(k_r x) dx, \quad r = 1, \dots, R,$$

with a constant period. The solution is based on a previous design described in Brudaru [2], [3] and is obtained by adding some preprocessing systolic arrays to that network in order to express Fillon's formula as an inner product.

2. Description of the Algorithm

Following [1], Fillon's formula is

$$\int_a^b f(x) \sin kx dx \simeq I,$$

$$(1) \quad I = h[Af(a) \cos ka - Af(b) \cos kb + B \star S + C \star T],$$

where $2nh = b - a$, and

$$(2) \quad A \frac{1}{kh} + \frac{1}{2k^2 h^2} \sin 2kh - \frac{2}{k^3 h^3} \sin^2 kh,$$

$$(3) \quad B = \frac{1}{k^2 h^2} (1 + \cos^2 kh) - \frac{1}{k^3 h^3} \sin 2kh,$$

$$(4) \quad C = \frac{4}{k^3 h^3} \sin kh - \frac{4}{k^2 h^2} \cos kh,$$

$$(5) \quad S = -f(a) \sin ka - f(b) \sin kb + 2 \sum_{i=0}^n f(a + 2ih) \sin(ka + 2ikh),$$

$$(6) \quad T = \sum_{i=1}^n f(a + (2i - 1)h) \sin(ka + (2i - 1)kh).$$

By putting $\theta = kh$, $p = 1/\theta$, $s = \sin \theta$, $c = \cos \theta$ and $z = sp$, (2)-(4) become

$$(7) \quad A = a_0 + z(a_1 + a_2 z),$$

$$(8) \quad B = b_0 + z(b_1 + b_2 z),$$

$$(9) \quad C = c_0 + z(c_1 + c_2 z),$$

where

$$(10) \quad a_0 = p, \quad a_1 = pc, \quad a_2 = -2p,$$

$$(11) \quad b_0 = 2p^2, \quad b_1 = -2p^2 c, \quad b_2 = -1,$$

$$(12) \quad c_0 = -4cp^2, \quad c_1 = 4p^2, \quad c_2 = 0.$$

Now,

$$(13) \quad I = \sum_{j=1}^{2n+5} f(x_j) w_j,$$

where, the nodes are

$$(14) \quad \begin{aligned} x_1 &= a, & x_2 &= b, & x_3 &= a, & x_4 &= b, \\ x_{5+i} &= a + 2ih, & i &= 0, \dots, n, \end{aligned}$$

$$x_{5+n+i} = a + (2i - 1)h, \quad i = 1, \dots, n,$$

while the weights are

$$\begin{aligned}
 w_1 &= A \sin\left(\frac{\pi}{2} - ka\right), & w_2 &= A \sin\left(kb - \frac{\pi}{2}\right), \\
 w_3 &= B \sin(-ka), & w_4 &= B \sin(-kb), \\
 w_{5+i} &= 2B \sin(ka + 2ikh), & i &= 0, \dots, n, \\
 w_{5+n+i} &= C \sin(ka + (2i-1)kh), & i &= 1, \dots, n.
 \end{aligned}
 \tag{15}$$

3. Mapping the Algorithm onto Hardware

Since the basic network (BN) described in Brudaru [3, §3] is able to compute (13) only if the weights are sent from the host computer, we shall focus on the systolic computation of the weights defined by (15) inside of the network.

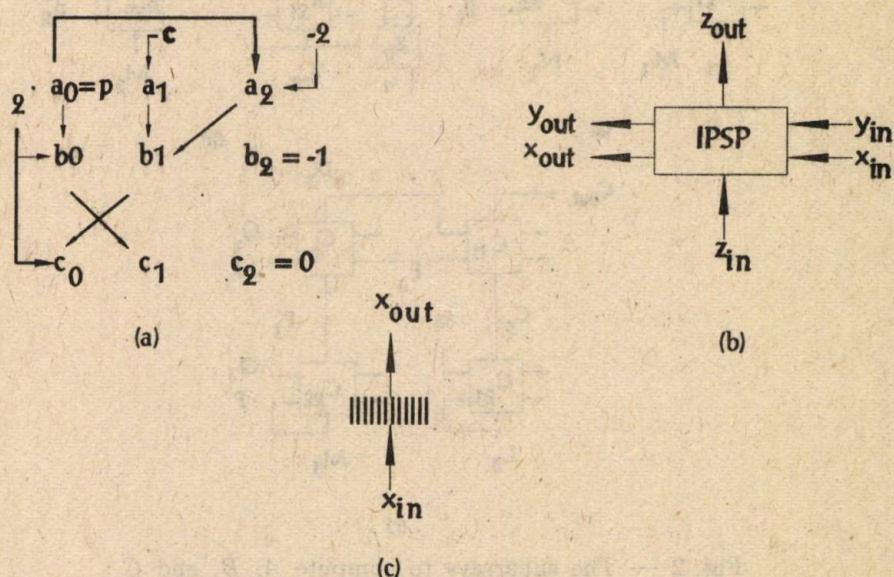


Fig. 1 — The precedence constraints (a), IPSP (b), 1 CT delay processor (c).

Further, the clock tick (CT) represents the time to perform a division or both a multiplication and an addition. The variable t denotes the time which is a count of the number of CTs. We suppose that each processor is active during each CT. If L is the label of a pin in the network then $L(t)$ designates the value circulating through this port during the t -th CT. Since the period of BN is 1 CT, our goal is to ensure that all preprocessing arrays, have 1 CT as period.

Let us implement the computation of A , B and C . From (10)–(12) we obtain the digraph of dependencies depicted in Fig. 1(a). From (7)–(9) it results that the computation of A , B and C can be regarded as a polynomial evaluation. To

do this evaluation, we use an inner product step processor (IPSP) depicted in Fig. 1(b) which works so that $z_{out}(t+1) = z_{in}(t) + x_{in}(t)y_{in}(t)$, $x_{out}(t+1) = x_{in}(t)$ and $y_{out}(t+1) = y_{in}(t)$, $t \geq 0$.

The processor in Fig. 1(c) is a 1 CT delay processor. Further, denote by $d(L, L') = r$ the fact that L is connected to L' through r delay processors.

3.1. The Subarrays to Compute A , B and C

The systolic subarrays to compute A , B and C are depicted in Fig. 2(a, b, c), respectively. Let us examine the way to synchronize the activity of these arrays with I/O operations.

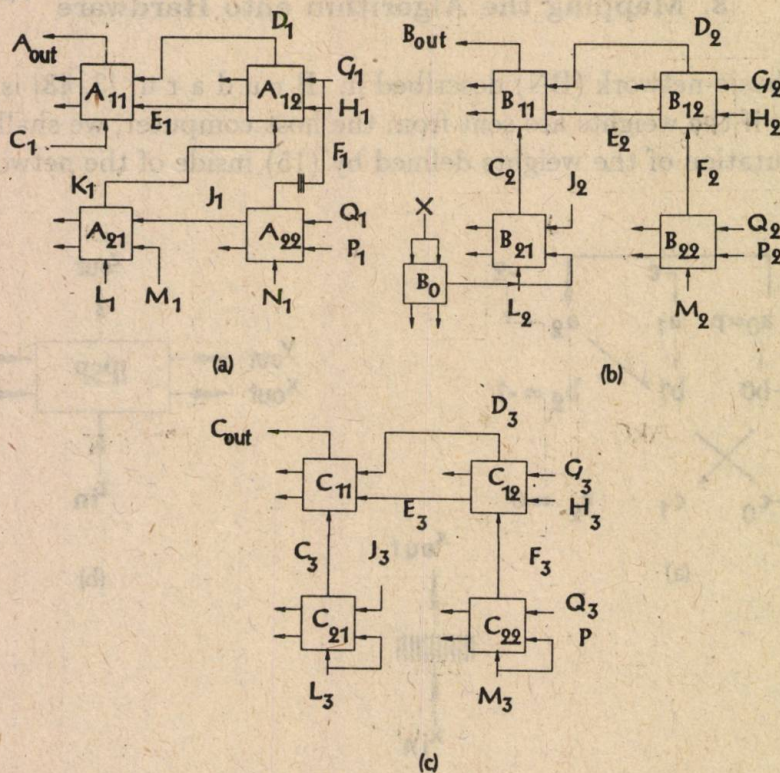


Fig. 2 — The subarrays to compute A , B , and C .

Suppose that $A_{out}(t_A) = A$. It results that $C_1(t_A - 1) = a_0 = p$, $E_1(t_A - 1) = z$ and $D_1(t_A - 1) = a_1 + a_2z$. Therefore $K_1(t_A - 2) = a_1$, $H_1(t_A - 2) = z$ and $G_1(t_A - 2) = F_1(t_A - 3) = a_2$.

We have that A_{21} and A_{22} requires $L_1(t_A - 3) = 0$, $M_1(t_A - 3) = c$ and $J_1(t_A - 3) = p$. Consequently, $Q_1(t_A - 4) = p$, $P_1(t_A - 4) = -2$ and $N_1(t_A - 4) = 0$. Now, if $B_{out}(t_B) = B$, we obtain $C_2(t_B - 1) = b_0$, $E_2(t_B - 1) = z$ and $D_2(t_B - 1) = b_1 + b_2z$. Thus $L_2(t_B - 2) = p_2$, $J_2(t_B - 2) = 1$ and $X(t_B - 3) = p$.

On the other hand, it results that $G_2(t_B - 2) = -1$, $H_2(t_B - 2) = z$ and $F_2(t_B - 2) = b_1$. Consequently, $N_2(t_B - 3) = 0$, $Q_2(t_B - 3) = a_1$ and $P_2(t_B - 3) = a_2$, and because $F_1(t_A - 3) = a_2$ and $K_1(t_A - 2) = a_1$, it results $t_A - 2 \leq t_B - 3$, $t_A - 3 \leq t_B - 3$,

i.e. $t_B \geq t_A + 1$. In order to save delay processors, we take $t_B = t_A + 1$ and therefore $d(K_1, Q_2) = 0$ and $d(F_1, P_2) = 1$.

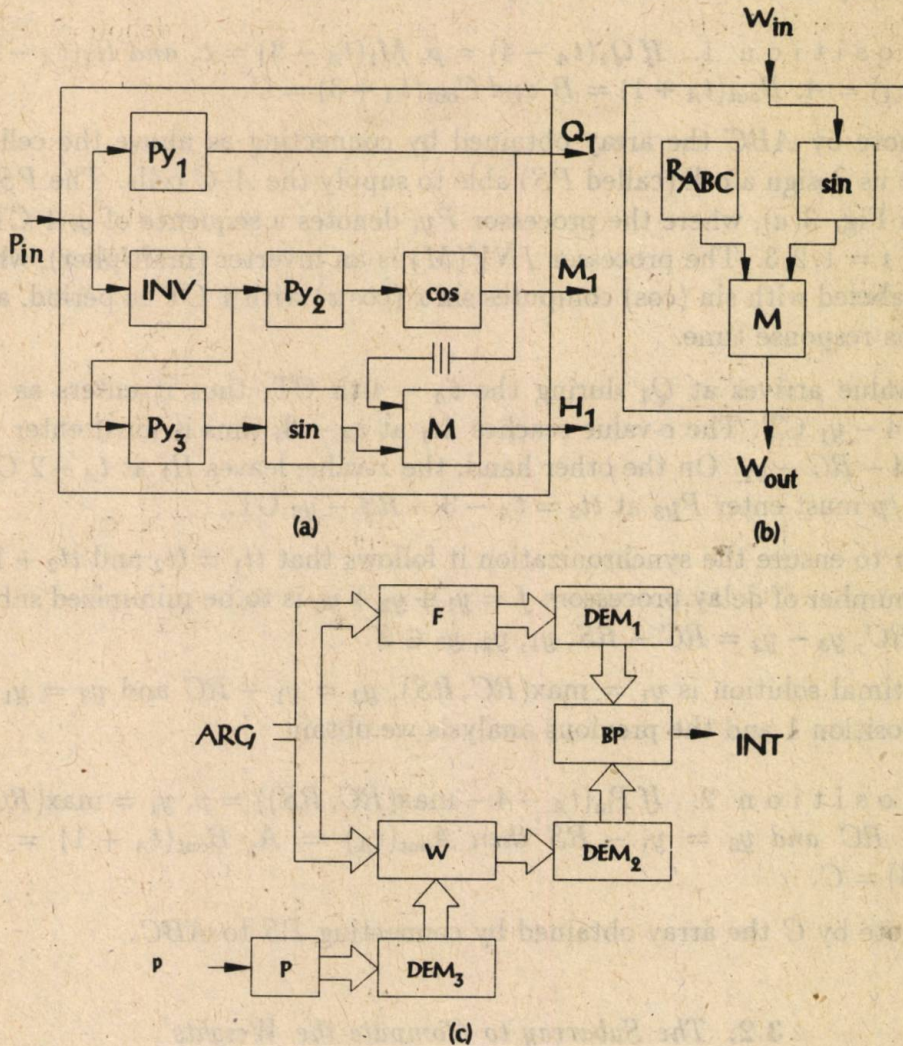


Fig. 3 — PS-cell (a), BW-cell (b), the block diagram of the network implementing Fillon's formula (c).

Finally, if $C_{out}(t_C) = C$, then $C_3(t_C - 1) = c_0$, $E_3(t_C - 1) = z$ and $D_3(t_C - 1) = c_1 + c_2z$. Therefore, $J_3(t_C - 2) = 1$ and $L_3(t_C - 2) = b_1$. But $F_2(t_B - 2) = b_1$ thus $t_B - 2 \leq t_C - 2$. On the other hand, we obtain $H_3(t_C - 2) = z$, $G_3(t_C - 2) = 0$ and $F_3(t_C - 2) = c_1 = 2b_0$ and therefore $Q_3(t_C - 3) = 1$ and $N_3(t_C - 3) = b_0$. Since $C_2(t_B - 1) = b_0$, we need $t_B - 1 \leq t_C - 3$ and from the above inequality we obtain $t_C \geq t_B + 2$. We can take $t_C = t_B + 2$ and consequently $d(F_2, L_3) = 2$ and $d(C_2, N_3) = 0$.

If a given piece of data must enter many times, we can send it along a hamiltonian path passing through the inputs requiring this data. Consequently, we obtain $d(Q_1, X) = 2$, $d(X, C_1) = 1$, $d(H_1, H_2) = 1$, $d(H_2, H_3) = 2$, $d(N_1, L_1) = 1$,

$d(L_1, G_3) = 4$, $d(J_2, Q_3) = 1$ and $d(Q_3, J_3) = 1$. We suppose that all constants are loaded in some registers by a reset command.

As a consequence of the above analysis, we can state

Proposition 1. *If $Q_1(t_A - 4) = p$, $M_1(t_A - 3) = c$, and $H_1(t_A - 2) = z$, then $A_{out}(t_A) = A$, $B_{out}(t_A + 1) = B$ and $C_{out}(t_A + 3) = C$.*

We denote by ABC the array obtained by connecting as above the cells A , B and C . Let us design a cell (called PS) able to supply the $A-C$ cells. The PS -cell is depicted in Fig. 3(a), where the processor Py_i denotes a sequence of y_i 1 CT-delay processors, $i = 1, 2, 3$. The processor $INV(M)$ is an inverter (multiplier), while the processor labeled with $\sin$ ($\cos$) computes $\sin x$ ($\cos x$) with 1 CT as period, and has RS (RC) as response time.

The p -value arrives at Q_1 during the $t_A - 4$ -th CT, thus it enters as Py_1 at $tt_1 = t_A - 4 - y_1$ CT. The c -value reaches M_1 at $t_A - 3$, thus it must enter INV at $tt_2 = t_A - 4 - RC - y_2$. On the other hand, the z -value leaves H_1 at $t_A - 2$ CT, and therefore $1/p$ must enter Py_3 at $tt_3 = t_A - 3 - RS - y_3$ CT.

In order to ensure the synchronization it follows that $tt_1 = tt_2$ and $tt_2 + 1 = tt_3$. The total number of delay processors $f = y_1 + y_2 + y_3$ is to be minimized subject to $y_1 - y_2 = RC$, $y_3 - y_2 = RC - RS$, $y_1, y_2, y_3 \in \mathbb{Z}$.

The optimal solution is $y_1 = \max(RC, RS)$, $y_2 = y_1 - RC$ and $y_3 = y_1 - RS$. From Proposition 1 and the previous analysis we obtain

Proposition 2. *If $P_{in}(t_A - 4 - \max(RC, RS)) = p$, $y_1 = \max(RC, RS)$, $y_2 = y_1 - RC$ and $y_3 = y_1 - RS$ then $A_{out}(t_A) = A$, $B_{out}(t_A + 1) = B$ and $C_{out}(t_A + 3) = C$.*

We denote by C the array obtained by connecting PS to ABC .

3.2. The Subarray to Compute the Weights

The basic cell (BW) to construct the subarray computing the weights (15) is shown in Fig. 3(b). It works so that $w_{out}(t + RS + 1) = \sin(w_{in}(t))R_{ABC}(t)$, where $R_{ABC}(t)$ designates the contents of the register R_{ABC} during the t -th CT.

3.3. The Entire Network

The block diagram of the entire network implementing Fillon's method is given in Fig. 3(c). The subnetwork denoted by F accomplishes the computation of f_r , $r = 1, \dots, R$. The nodes are received through the ARG -input.

The values of the functions go through the demultiplexor $D \in M_1$ to the basic pipe (BP) computing the integrals. On the other hand, the P -cell for each $r \in \{1, \dots, R\}$ (via the demultiplexor $D \in M_3$) sends the values of A , B and C to the W -subnetwork containing R copies of the BW -cell. W receives the arguments and computes the

weights. The weights are pumped through $D \in M_2$ to BP . The integrals are extracted from INT. The period of the network is 1 CT.

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REFERENCES

1. Scheid F., *Theory and Problems of Numerical Analysis*. Schaum's Outline Series, McGraw-Hill Book Company, New York, 1968.
2. Brudaru O., *A Systolic Array for Numerical Integration by Using the Trapezoidal and Simpson Formulas*. Studia Universitatis Babeș-Bolyai, Cluj-Napoca, Mathematica, **XXXIII**, 3, 27-31 (1988).
3. Brudaru O., *A Systolic Network for Numerical Integration*. Studia Universitatis Babeș-Bolyai, Cluj-Napoca, Mathematica, **XXXIV**, 4, 63-68 (1989).

INTEGRAREA FUNCȚIILOR OSCILANTE CU REȚELE SISTOLICE

(Rezumat)

Se prezintă o implementare sistolică a formulei lui Fillon pentru integrarea funcțiilor oscilante. Arhitectura sistolică propusă are perioada egală cu un tact.

weights. The weights and standard deviations of the network are extracted from the network. The network is a CT.

Thompson, J. A. (1957)
Statistical Analysis of Data

Thompson, J. A. (1957)

REFERENCES

1. H. A. T. (1957) and J. A. T. (1957) in "Statistical Analysis of Data" (Thompson, J. A., ed.), New York, 1957.
2. H. A. T. (1957) and J. A. T. (1957) in "Statistical Analysis of Data" (Thompson, J. A., ed.), New York, 1957.
3. H. A. T. (1957) and J. A. T. (1957) in "Statistical Analysis of Data" (Thompson, J. A., ed.), New York, 1957.
4. H. A. T. (1957) and J. A. T. (1957) in "Statistical Analysis of Data" (Thompson, J. A., ed.), New York, 1957.

STATISTICAL ANALYSIS OF DATA

(Thompson)

The present statistical analysis is based on the data obtained from the network. The data are analyzed and the results are presented in the following table.

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CR-SUBMANIFOLDS NORMAL TO THE STRUCTURE VECTOR FIELD OF A TRANS-SASAKIAN MANIFOLD

BY

CĂLIN AGUT

Abstract. One proves that the CR-submanifold normal to the structure vector field of a trans-Sasakian manifold are anti-invariant submanifold when $\alpha \neq 0$.

Key words: CR-submanifold, trans-Sasakian manifold, anti-invariant submanifold.

Let $\tilde{M}(\varphi, \xi, \eta, g)$ be a $2n+1$ -dimensional almost contact metric manifold, that is, φ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is a Riemannian metric, satisfying the following relations (cf. Blair [3]):

$$(1) \quad \varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0,$$

and

$$(2) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \Gamma(TM)$.

In 1985 Oubina [5] introduced the special class of trans-Sasakian manifolds as follows. Denote by $\tilde{\nabla}$ the Levi-Civita connection on $\tilde{M}$ with respect to g . Then the almost contact manifold $\tilde{M}$ is said to be a *trans-Sasakian manifold* if there exist two smooth functions α and β on $\tilde{M}$ such that the covariant derivative of φ with respect to $\tilde{\nabla}$ is given by

$$(3) \quad (\tilde{\nabla}_X \varphi)Y = \alpha \{g(X, Y)\xi - \eta(Y)X\} + \beta \{g(\varphi X, Y)\xi - \eta(Y)\varphi X\},$$

for any $X, Y \in \Gamma(TM)$. In particular, for $\alpha = 1$ and $\beta = 0$ it follows that $\tilde{M}$ is a Sasakian manifold and for $\alpha = 0$ and $\beta = 1$ we obtain a Kenmotsu manifold (cf. Yano - Kon [7]). Moreover, the covariant derivative of the structure vector field is given by

$$(4) \quad \tilde{\nabla}_X \xi = -\alpha \varphi X + \beta \{X - \eta(X)\xi\},$$

for any $X \in \Gamma(TM)$.

Let M be an m -dimensional submanifold of $\tilde{M}$ such that ξ is normal to M . Then according to the general concept of CR-submanifold introduced by Bejancu [1] we give the following

Definition. The submanifold M is said to be a *CR-submanifold* if the tangent bundle of M has the decomposition $TM = D \oplus D^\perp$, where D and $D^\perp$ are two orthogonal distributions on M such that $\varphi(D) \subset D$ and $\varphi(D^\perp) \subset TM^\perp$. This concept has been studied by Fofana [4] and Shahid [6]. If in particular, $D = \{0\}$ then M is called an *anti-invariant submanifold*.

The main purpose of the paper is to prove the following result.

Theorem 1. *Let M be a CR-submanifold normal to the structure vector field of a trans-Sasakian manifold $\tilde{M}$ with $\alpha \neq 0$. Then M is an anti-invariant submanifold.*

Proof. Let X be a vector field on M . Then

$$\eta(X) = g(X, \xi) = 0,$$

and (4) becomes

$$(5) \quad \tilde{\nabla}_X \xi = -\alpha \varphi X + \beta X.$$

Then, taking into account that $\tilde{\nabla}$ is a Riemannian connection and by using (5) we obtain

$$(6) \quad g(\xi, \tilde{\nabla}_X Y) = -g(\tilde{\nabla}_X \xi, Y) = \alpha g(\varphi X, Y) - \beta g(X, Y),$$

for any $X, Y \in \Gamma(D)$. Since $\tilde{\nabla}$ is torsion-free and g is symmetric, by using (6) and (2) we deduce

$$g(\xi, [X, Y]) = 2\alpha g(\varphi X, Y).$$

As $[X, Y]$ is tangent to M while ξ is normal to M and $\alpha \neq 0$ we derive $g(\varphi X, Y) = 0$. Replace Y by φY and by using (2) we conclude $g(X, Y) = 0$. Hence $D = \{0\}$ and thus the CR-submanifold M must be anti-invariant. The proof is complete.

As CR-submanifolds normal to ξ in a trans-Sasakian manifold $\tilde{M}$ are anti-invariant provided $\alpha \neq 0$, we close the paper with some general results on the second fundamental form of an anti-invariant submanifold of $\tilde{M}$.

First we recall (see Bejancu [2]) that the Gauss and Weingarten equations of M in $\tilde{M}$ are as follows:

$$(7) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

and

$$(8) \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad \forall X \in \Gamma(TM), \quad N \in \Gamma(TM^\perp),$$

where ∇ is the Levi-Civita connection on M with respect to the induced metric g on M , h is the second fundamental form of M , A_N is the Weingarten operator and $\nabla^\perp$ is the normal connection of M . We have to note that h and A_N are related by

$$(9) \quad g(h(X, Y), N) = g(A_N X, Y), \quad \forall X, Y \in \Gamma(TM), \quad N \in \Gamma(TM^\perp).$$

Next, replace N by ξ in (8) and then comparing (8) and (5) we obtain

$$(10) \quad A_\xi X = -\beta X, \quad \forall X \in \Gamma(TM).$$

Theorem 2. *Let M be an anti-invariant submanifold normal to ξ and with parallel second fundamental form in a trans-Sasakian manifold $\tilde{M}$. Then we have the following assertions:*

i) *If $\alpha = 0$ then β must be a constant on M .*

ii) *If $\alpha \neq 0$ then*

$$(11) \quad A_{\varphi X} Y = \frac{X(\beta)}{\alpha} Y, \quad \forall X, Y \in \Gamma(TM).$$

Proof. Taking into account that h is parallel and ∇ is a metric connection, and by using (9) and (10) we obtain

$$(12) \quad 0 = g((\nabla_X h)(Y, Z), \xi) = g(\nabla_X^\perp h(Y, Z), \xi) + \beta X(g(Y, Z)),$$

for any $X, Y, Z \in \Gamma(TM)$. On the other hand, by using (5) and (9) we deduce

$$(13) \quad X(g(h(Y, Z), \xi)) = g(\nabla_X^\perp h(Y, Z), \xi) - \alpha g(A_{\varphi X} Y, Z).$$

Due to (9) and (10) we infer

$$(14) \quad g(h(Y, Z), \xi) = -\beta g(Y, Z).$$

Hence (13) and (14) yields

$$(15) \quad g(\nabla_X^\perp h(Y, Z), \xi) = \alpha g(A_{\varphi X} Y, Z) - X(\beta g(Y, Z)).$$

Comparing (12) and (15) we derive

$$g(\alpha A_{\varphi X} Y - X(\beta)Y, Z) = 0,$$

which implies

$$(16) \quad \alpha A_{\varphi X} Y = X(\beta)Y,$$

since g is a Riemannian metric on M . Clearly both assertions of the theorem follow from (16).

From (10) and (11) we obtain the following

Theorem 3. *Let M be an anti-invariant submanifold normal to ξ and with parallel second fundamental form in a trans-Sasakian manifold $\tilde{M}$. Then we have the following assertions :*

- i) *If β is a non-null constant then M is never totally umbilical.*
- ii) *If $\beta = 0$ and $m = n$ then M is totally geodesic.*

The assertion ii) has been proved by Yano - Kon [7, p.345] for anti-invariant submanifolds normal to ξ in a Sasakian manifold.

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REFERENCES

1. Bejancu A., *CR-Submanifolds of a Kaehler Manifold (I)*. Proc. Amer. Math. Soc., **69**, 134-142 (1978).
2. Bejancu A., *Geometry of CR-Submanifolds*. D. Reidel, Dordrecht, 1986.
3. Blair D. E., *Contact Manifolds in Riemannian Geometry*. Lecture Notes Math., **509**, Springer, Berlin, 1976.
4. Fofana L., *Geometry of CR-Submanifolds*. Ph. D. Thesis, Univ. Bucharest, 1996.
5. Oubiña J. A., *New Class of a Almost Contact Metric Structure*. Publ. Math. Debrecen, **32**, 187-193 (1985).
6. Shahid M. H., *CR-Submanifolds of a Trans-Sasakian Manifold*. Indian J. Pure Appl. Math., **22**, 1007-1012 (1991).
7. Yano K., Kon M., *Structures of Manifolds*. World Scientific, Singapore, 1984.

CR-SUBVARIETĂȚI NORMALE CÂMPULUI VECTORIAL DE STRUCTURĂ ALE UNEI VARIETĂȚI TRANS-SASAKI

(Rezumat)

Se demonstrează că CR-subvarietățile normale câmpului vectorial de structură ale unei varietăți trans-Sasaki sunt subvarietăți anti-invariante în cazul în care $\alpha \neq 0$. Se obțin apoi proprietățile fundamentale ale subvarietăților anti-invariante cu forma a doua fundamentală paralelă.

QUELQUES THÉORÈMES DE POINT FIXE POUR DES PAIRES D'APPLICATIONS MULTIVOQUES

PAR

NICOLETA NEGOESCU

Abstract. Common fixed point theorems for pairs of multivalued mappings defined on complete metrical spaces as well as of compact metrical spaces are proved.

Key words: fixed point theorem, multivalued mapping, metrical spaces.

1. Dans cette Note nous démontrons d'abord des théorèmes de point fixe commun pour des paires d'applications multivoques définies sur des espaces métriques complets (qui satisfont aux inégalités contractives (1) et (2)) et puis des théorèmes de point fixe pour des paires d'applications multivoques définies sur des espaces métriques compacts (qui satisfont aux inégalités contractives (3) et (4) et aux inégalités expansives (5), (6)).

L'idée d'étudier ce genre de problèmes pour des paires d'applications multivoques nous a été suggérée par les résultats intéressants, quelques uns très récents, obtenus par B. Fisher [1], T. Kubiak [4], F. Zalar, R. Rehman and B. Ahmad [2], R. K., Jain et S. P. Dixit [3], B. G. Patchpatte [9], B. E. Rhoades and B. Watson [10], T. Taniguchi [12] et Nicoleta Negoescu [6]...[8] pour d'autres conditions contractives ou expansives ou pour des opérateurs définis sur des espaces d'un autre type.

2. Nous rappelons d'abord quelques notions et résultats qui nous sont nécessaires.

Soient (X, d) un espace métrique et A, B deux sousensembles nonvides de X . Alors on définit:

$$D(x, A) = \inf \{d(x, a) : a \in A, x \in X, x \text{ fixé}\},$$

$$\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\},$$

$$H(A, B) = \max \left\{ \sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A) \right\},$$

(où A ou B peut être un ensemble formé d'un seul point),

$$BN(X) = \{A : A \subset X, A \neq \emptyset, A \text{ borné}\},$$

$$CL(X) = \{A : A \subset X, A \neq \emptyset, A \text{ fermé}\},$$

$$C(X) = \{A : A \subset X, A \neq \emptyset, A \text{ compact}\},$$

$$CB(X) = BN(X) \cap CL(X).$$

Observations. 1) La fonction D est continue (v. [5]).

2) Évidemment on a $D(x, A) \leq \delta(x, A)$, $\delta(A, B) \geq H(A, B)$, $d(x, y) \leq \delta(A, B)$ et $d(x, y) \leq H(A, B)$ pour tous $x \in A$ et $y \in B$.

3) La fonction H est une métrique sur $CB(X)$, appelée la métrique de Hausdorff (et sur $CL(X)$ si X est un espace compact, v. [4] [5]).

Le point $x \in X$ est point fixe de l'application multivoque $S : X \rightarrow CB(X)$ si $x \in Sx$ et $x \in X$ est un point fixe commun des applications multivoques $S, T : X \rightarrow CB(X)$ si $x \in Sx \cap Tx$.

3. Théorème 1. Soient (X, d) un espace métrique complet et $S, T : X \rightarrow CB(X)$ deux applications multivoques qui satisfont à l'inégalité

$$(1) \quad H^2(Sx, Tx) \leq a^2 D(x, Sx) D(y, Ty) + b D(x, Ty) D(y, Sx)$$

pour tous $x, y \in X$ où $0 < a < 1, b \in \mathbb{R}_+$. Alors S, T ont un point fixe commun $x \in X$. Pour $b < 1$ point fixe x est unique.

Démonstration. Soient $x_0 \in X$ et $x_1 \in Sx_0$ arbitraires. Si $x_0 = x_1$ on a $x_0 \in Sx_0$ et alors $D^2(x_0, Tx_0) \leq H^2(Sx_0, Tx_0) \leq a^2 D(x_0, Sx_0) D(x_0, Tx_0) + b D(x_0, Tx_0) D(x_0, Sx_0)$ ou $D^2(x_0, Tx_0) \leq 0$. Donc $D^2(x_0, Tx_0) = 0$ et $x_0 \in Tx_0$ (l'ensemble Tx_0 est fermé).

On construit maintenant la suite (x_n) de la manière suivante: on suppose $x_n \neq x_{n+1}, \forall n \in \mathbb{N}$; $x_0 \in X$ (x_0 arbitraire) et $x_1 \in Sx_0$. On choisit $x_2 \in Tx_1$ tel que $d(x_1, x_2) \leq a^{-1/2} H(Sx_0, Tx_1)$. Alors

$$\begin{aligned} d^2(x_1, x_2) &\leq \left(\frac{1}{\sqrt{a}}\right)^2 H^2(Sx_0, Tx_1) \leq \frac{a^2}{a} D(x_0, Sx_0) D(x_1, Tx_1) + \\ &+ \frac{b}{a} D(x_0, Tx_1) D(x_1, Sx_0) < ad(x_0, x_1) d(x_1, x_2) \end{aligned}$$

(car $x_1 \in Sx_0$) ou $d(x_1, x_2) \geq ad(x_0, x_1)$ (car $x_1 \neq x_2$).

En général on obtient $d(x_{n+1}, x_{n+2}) \leq ad(x_n, x_{n+1})$ pour $n = 1, 2, \dots$, où $x_{2n-1} \in Sx_{2n-2}$ et $x_{2n} \in Tx_{2n-1}$ sont tels que

$$d(x_{2n-1}, x_{2n}) \leq \frac{1}{\sqrt{a}} H(Sx_{2n-2}, Tx_{2n-1}), \quad x_{2n-1} \neq x_{2n}, \quad \forall n \in \mathbb{N},$$

et donc $d(x_n, x_{n+1}) \leq a^n d(x_0, x_1)$, $\forall n \in \mathbb{N}$. Mais $0 < a < 1$ et alors (x_n) est une suite de Cauchy. L'espace (X, d) étant complet, il suit que la suite (x_n) est convergente à un point $x \in X$. Alors

$$D(x, Sx) \leq D(x, x_{2n}) + D(x_{2n}, Sx) \leq D(x, x_n) + H(Tx_{2n-1}, Sx)$$

et, en employant la condition (1) et en faisant $x_n \rightarrow x$ pour $n \rightarrow \infty$ dans la dernière inégalité, on a $D(x, Sx) \leq 0$. Il suit que $D(x, Sx) = 0$ ou $x \in Sx$.

De la même manière on peut montrer que $x \in Tx$ et donc $x \in Sx \cap Tx$.

Pour démontrer l'unicité du point $x \in X$, on suppose qu'ils existent deux points fixes communs différents $x, y \in X$, $x \neq y$, de S et T . Alors $d^2(x, y) \leq H^2(Sx, Ty) \leq a^2 D(x, Sx)D(y, Ty) + bD(x, Ty)D(y, Sx)$, donc $d^2(x, y) \leq bd^2(x, y)$ ou $(1-b)d^2(x, y) \leq 0$. Il suit que seulement pour $b < 1$ le point fixe commun de S, T est unique.

On peut démontrer aussi, d'une manière similaire avec la démonstration du Théorème 1, le suivant

Théorème 2. Soient (X, d) un espace métrique complet et $S, T : X \rightarrow BN(X)$ deux applications multivoques qui satisfont à l'inégalité

$$(2) \quad \delta^2(Sx, Tx) \leq a^2 H(x, Sx)H(y, Ty) + bD(x, Ty)D(y, Sx),$$

pour $\forall x, y \in X$, $0 < a < 1$, $b \in \mathbb{R}_+$.

Alors S et T ont un point fixe commun, qui est un point fixe unique pour $b < 1$.

Observations. 1) Le Théorème 1 est vrai aussi si on remplace la condition (1) par une des inégalités suivantes:

$$H^2(Sx, Ty) \leq a^2 \frac{D^2(x, Sx)D^2(y, Ty)}{D(x, Sx)D(y, Ty) + D(y, Ty)D(x, Sx)}, \quad \forall x, y \in X, \quad 0 < a < 1;$$

$$H^2(Sx, Ty) \leq a^2 \max \{d^2(x, y), D(x, Sx)D(y, Ty), D(x, Ty)D(y, Sx)\}$$

pour tous $x, y \in X$, $0 \leq a < 1$ (v. [7]);

$$H^2(Sx, Ty) \leq a^2 D^2(x, Sx)D(y, Ty) + bD(y, Sx)[D(x, Sx)D(y, Ty) + D(x, Ty)]$$

pour tous $x, y \in X$, $0 < a < 1$, $b \in \mathbb{R}_+$ ou

$$H^{n+1}(Sx, Ty) \leq a^2 D^n(x, Sx)D(y, Ty) + bD^n(x, Ty) + D(y, Sx),$$

$$\forall x, y \in X, \quad n = 2, 3, \dots \quad 0 < a < 1, \quad b \in \mathbb{R}_+.$$

2) On peut obtenir aussi des théorèmes analogues du Théorème 2 en remplaçant la condition (2) par des conditions ressemblantes avec celles de l'observation antérieure.

3) Si $b = 0$ dans les Théorèmes 1 et 2, alors on obtient les Théorèmes 1 et 2 de [2].

4) Les Théorèmes 1 et 2 sont vrais aussi pour une seule application multivoque $S = T : X \rightarrow CB(X)$, pour deux opérateurs $S, T : X \rightarrow X$ ou pour un seul opérateur.

5) Des Théorèmes 1 et 2 peut obtenir aussi les résultats suivants pour des espaces métriques compacts:

Si (X, d) est un espace métrique compact, $S, T : X \rightarrow CL(X)$, S ou T continue et la condition (1) (ou (2)) est satisfaite alors S ou T a un point fixe.

4. Dans ce qui suit nous présenterons encore quelques théorèmes de ce type, mais en employant une autre méthode de démonstration que celle des Théorèmes 1 et 2.

Théorème 3. Soient (X, d) un espace métrique compact $S, T : X \rightarrow CL(X)$ deux applications multivoques et S ou T est continue sur X . Nous supposons que pour tous $x, y \in X$, $x \neq y$, et $a \in \mathbb{R}_+$ ($a \in \mathbb{R}$, $a \geq 0$) on a

$$(3) \quad H^2(Sx, Ty) < D(x, Sx)D(y, Ty) + aD(x, Ty)D(y, Sx).$$

Alors S ou T a un point fixe $u \in X$.

Démonstration. Nous supposons que l'application multivoque S est continue et nous notons par $f(x) = D(x, Sx)$. Car f est une application continue sur X (D est continue et X est un espace compact (v. [5]) alors elle atteint son minimum dans un point $x_0 \in X$. Soit $f(x_0) = \min \{f(x) : x \in X\}$. Nous choisissons les points $x_1 \in Sx_0$, $x_2 \in Tx_1$ et $x_3 \in Sx_2$ tels que

$$d(x_0, x_1) = D(x_0, Sx_0), \quad d(x_1, x_2) = D(x_1, Tx_1) \quad \text{et} \quad d(x_2, x_3) = D(x_2, Sx_2).$$

Nous montrons qu'il est vraie une des égalités suivantes: $D(x_0, Sx_0) = 0$ ou $D(x_1, Tx_1) = 0$, c'est-à-dire $x_0 \in Sx_0$ ou $x_1 \in Tx_1$. Nous supposons que $D(x_0, Sx_0) > 0$ et $D(x_1, Tx_1) > 0$. Il suit que:

$$d^2(x_1, x_2) \leq H^2(Sx_0, Tx_1) < D(x_0, Sx_0)D(x_1, Tx_1) + aD(x_0, Tx_1)D(x_1, Sx_0)$$

ou $d^2(x_1, x_2) < d(x_0, x_1)d(x_1, x_2)$ (car $x_1 \in Sx_0$ et alors $D(x_1, Sx_0) = 0$).

Car $d(x_1, x_2) = D(x_1, Tx_1) > 0$, on a

$$(i) \quad d(x_1, x_2) < d(x_0, x_1).$$

De la même manière nous obtenons

$$d^2(x_2, x_3) \leq H^2(Sx_2, Tx_1) < D(x_2, Sx_2)D(x_1, Tx_1) + aD(x_2, Tx_1)D(x_1, Sx_2)$$

($x_2 \in Tx_1$ et $D(x_2, Tx_1) = 0$) et donc $d^2(x_2, x_3) < d(x_2, x_3)d(x_1, x_2)$. Nous avons deux possibilités: $D(x_2, Sx_2) = 0$ ou $D(x_2, Sx_2) > 0$. Si $D(x_2, Sx_2) = 0$, S a le point fixe x_2 et nous n'avons rien à démontrer. Si $d(x_2, x_3) = D(x_2, Sx_2) > 0$ il suit

$$(ii) \quad d(x_2, x_3) < d(x_1, x_2).$$

Des inégalités (i) et (ii) on obtient $d(x_2, x_3) < d(x_1, x_2) < d(x_0, x_1) = f(x_0)$ ce que contredit la minimalité de $f(x_0)$. Alors $D(x_0, Sx_0) = 0$ ou $D(x_1, Tx_1) = 0$;

ce que signifie que x_0 est un point fixe de S ($x_0 \in Sx_0$) ou x_1 est un point fixe de T ($x_0 \in Tx_0$).

Observations. 1) En procédant de la manière du Théorème 3 nous pouvons démontrer deux théorèmes analogues pour les applications multivoques S et T qui satisfont aux inégalités suivantes:

$$H^2(Sx, Ty) < \max \{D(x, Sx)D(y, Ty), D(x, Ty)D(y, Sx)\}, \quad \forall x, y \in X, \quad x \neq y;$$

$$H^2(Sx, Ty) < \frac{D^2(x, Sx)D^2(y, Ty)}{D(x, Sx)D(y, Ty) + D(x, Ty)D(y, Sx)}, \quad \forall x, y \in X, \quad x \neq y.$$

2) Si dans le Théorème 3 on prend $S = T, T : X \rightarrow X$ (X étant un espace compact) nous obtenons le résultat de B. F i s h e r [1]. Dans ce cas on peut montrer que le point fixe $u \in X$ de l'opérateur T est unique si $a \leq 1$.

3) Le Théorème 3 est vrai pour $S = T, T : X \rightarrow C(X)$, qui satisfait à l'inégalité suivante: pour tous $x, y \in X, x \neq y$ et $a \in \mathbb{R}_+$,

$$H^2(Tx, Ty) < D(x, Tx)D(y, Ty) + a[D(x, Ty)D(x, Tx) + D(x, Ty)D(y, Ty) + D(x, Ty)D(y, Tx)].$$

Le Théorème 3 est vrai aussi pour $S = T, T : X \rightarrow X$, qui satisfait l'inégalité antérieure (évidemment dans ce cas $d(x, Sx) = D(x, Sx)$) et on peut montrer que l'opérateur T a un seul point fixe si $a \leq 1$ (v. [3], [7], [8]).

4) On peut démontrer (de la même manière) un théorème analogue du Théorème 3 pour une seule application multivoque $T : X \rightarrow CL(X)$ en remplaçant l'inégalité (3) par l'inégalité suivante:

$$H^2(Tx, Ty) < a[D(x, Tx)D(y, Ty) + D(x, Ty)D(y, Tx)] + b[D(x, Tx)D(y, Tx) + D(x, Ty)D(y, Tx)] + c[D(x, Tx)D(y, Tx) + D(x, Ty)D(y, Ty)]$$

pour tous $x, y \in X, x \neq y$ et $a, b, c \geq 0, a + 2b + 2c \leq 1$ (v. [3], [7], [8]). Si dans l'inégalité antérieure on prend $S = T : X \rightarrow X$ et $b = 0$, alors on obtient le résultat de G. P a t c h p a t t e [9].

Maintenant on peut démontrer le suivant

T h é o r è m e 4. Soient (X, d) un espace métrique compact, $S, T : X \rightarrow CL(X)$ deux applications multivoques, S ou T continue, qui satisfont à l'inégalité

$$(4) \quad \delta^2(Sx, Ty) < H(x, Sx)H(y, Ty) + aD(x, Ty)D(y, Sx)$$

pour tous $x, y \in X, x \neq y$ et $a \in \mathbb{R}_+$.

Alors S ou T a un point fixe $u \in X$ tel que $Su = \{u\}$ ou $Tu = \{u\}$.

D é m o n s t r a t i o n. Soient l'application multivoque S continue et $f : X \rightarrow \mathbb{R}_+, f(x) = H(x, Sx)$. Donc f atteint son minimum dans un point $x_0 \in$

$\in X$, $(f(x_0) = \min \{f(x) : x \in X\})$. Nous choisissons $x_1 \in Sx_0$, $x_2 \in Tx_1$ et $x_3 \in Sx_2$ tels que

$$d(x_0, x_1) = H(x_0, Sx_0), \quad d(x_1, x_2) = H(x_1, Tx_1), \quad d(x_2, x_3) = H(x_2, Sx_2).$$

Nous montrons qu'il y a au moins une des égalités suivantes: $H(x_0, Sx_0) = 0$ ou $H(x_1, Tx_1) = 0$. Nous supposons que $H(x_0, Sx_0) > 0$ et $H(x_1, Tx_1) > 0$. Alors il suit

$$d^2(x_1, x_2) \leq \delta^2(Sx_0, Tx_1) < H(x_0, Sx_0)H(x_1, Tx_1) + aD(x_0, Tx_1)D(x_1, Sx_0).$$

$D(x_1, Sx_0) = 0$ car $x_1 \in Sx_0$ et donc $d^2(x_1, x_2) < d(x_1, x_2)d(x_0, x_1)$. Car $d(x_1, x_2) = H(x_1, Tx_1) > 0$, il suit

$$(i) \quad d(x_1, x_2) < d(x_0, x_1).$$

De la même manière il suit $d^2(x_2, x_3) \leq \delta^2(Sx_2, Tx_1) < H(x_2, Sx_2)H(x_1, Tx_1) + aD(x_2, Tx_1)D(x_1, Sx_2)$ et nous avons deux possibilités: $H(x_2, Sx_2) = 0$ ou $d(x_2, x_3) = H(x_2, Sx_2) > 0$. Si $H(x_2, Sx_2) = 0$, S a le point fixe x_2 et la démonstration est finie. Si $d(x_2, x_3) = H(x_2, Sx_2) > 0$, il suit

$$(ii) \quad d(x_2, x_3) < d(x_1, x_2).$$

Des inégalités (i) et (ii) il suit $d(x_2, x_3) < d(x_1, x_2) < d(x_0, x_1)$ ou $H(x_2, Sx_2) < H(x_1, Tx_1) < H(x_0, Sx_0)$, ce que contredit la définition de $f(x_0)$. Donc il suit que $H(x_0, Sx_0) = 0$ ou $H(x_1, Tx_1) = 0$, c'est-à-dire $Sx_0 = \{x_0\}$ et $Tx_1 = \{x_1\}$.

Observation. Le Théorème 4 est vrai aussi pour une seule application multivoque $S = T : X \rightarrow CL(X)$, pour deux opérateurs $S, T : X \rightarrow X$ ou pour un seul opérateur.

5. Maintenant nous présentons deux théorèmes de point fixe (analogues des Théorèmes 3 et 4), pour des applications multivoques expansives (v. [12] et [6]).

T h é o r è m e 5. Soient (X, d) un espace métrique compact et $S, T : X \rightarrow CL(X)$ deux applications multivoques et au moins une d'elles est continue sur X . Si pour tous $x, y \in X$, $x \neq y$ et $a \in \mathbb{R}_+$ est vraie l'inégalité

$$(5) \quad H^2(Sx, Ty) > D(x, Sx)D(y, Ty) + aD(x, Ty)D(y, Sx),$$

alors S ou T a un point fixe $u \in X$.

D é m o n s t r a t i o n. Nous supposons que l'application multivoque S est continue et nous notons par $f(x) = D(x, Sx)$. Parce que la fonction f est continue sur X (D est continue et X est un espace compact (v. [5])), alors la fonction f est bornée sur X et f atteint son maximum dans un point $x_0 \in X$. Soit $f(x_0) = \max \{f(x) : x \in X\}$. La démonstration suit de la même manière que la démonstration du Théorème 3, mais en employant la condition (5).

T h é o r è m e 6. Soient (X, d) un espace métrique compact et $S, T : X \rightarrow CL(X)$ deux applications multivoques, S ou T étant continue sur X , qui satisfont à l'inégalité

$$(6) \quad \delta^2(Sx, Ty) > H(x, Sx)H(y, Ty) + aD(x, Ty)D(y, Sx),$$

pour tous $x, y \in X$, $x \neq y$ et $a \in \mathbb{R}_+$.

Alors S ou T a un point tel que $Su = \{u\}$ ou $Tu = \{u\}$.

La démonstration du Théorème 6 suit de la même manière que la démonstration du Théorème 4, mais en employant la condition (6).

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BIBLIOGRAPHIE

1. Fisher B., *Fixed Point and Constants Mappings on Metric Spaces*. Rend. Accad. Lincei, **61**, 329-332 (1976).
2. Fazal-ur Rehman, Ahmad B., *Some Fixed Point Theorems in Complete Metric Spaces*. Math. Japonica, **36**, 239-243 (1991).
3. Jain R. K., Dixit S. P., *Some Fixed Point Theorems for Mappings in Pseudocompact Tihonov Space*. Publ. Math. Debrecen, **33**, 3-4, 195-197 (1986).
4. Kubiak T., *Fixed Point Theorems for Contractive Type Multivalued Mappings*. Math. Japonica, **30**, 1, 89-101 (1985).
5. Kubiakzyk I., *Some Fixed Point Theorems*. Demonstratio Math., **6**, 507-515 (1976).
6. Negoescu Nicoleta, *Quelques théorèmes de point fixe pour des applications expansives*. Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj, 1990, pp. 189-192.
7. Negoescu Nicoleta, *Observations sur des paires d'applications multivoques d'un certain type de contractivité*. Bul. Inst. Polit. Iași, **XXXV (XXXIX)**, 1 - 4, s. I, 21-25 (1989).
8. Negoescu Nicoleta, *Sur l'existence des points fixes des opérateurs définis sur des espaces pseudocompacts*. Bul. Inst. Polit. Iași, **XXXIX (XLIII)**, 1 - 4, s. I, 67-72 (1993).
9. Patchpatte B. G., *On Certain Fixed Points Mappings in Metric Spaces*. Jour. of M. A. C. T., **13**, 59-63 (1980).
10. Rhoades B. E., Watson B., *Fixed Points for Set-Valued Mappings on Metric Spaces*. Math. Japonica, **35**, 735-743 (1990).
11. Rus I. A., *Principii și aplicații ale teoriei punctului fix*. Ed. "Dacia", Cluj-Napoca, 1979.
12. Taniguchi T., *Common Fixed Points Theorems on Expansion Type Mappings on Complete Metric Spaces*. Math. Japonica, **34**, 139-142 (1989).

CÂTEVA TEOREME DE PUNCT FIX PENTRU PERECHI DE APLICAȚII MULTIVOCE

(Rezumat)

Se demonstrează mai întâi teoreme de punct fix comun pentru perechi de aplicații multivoce definite pe spații metrice complete (și care satisfac inegalitățile contractive (1) și (2)) și apoi teoreme de punct fix pentru perechi de aplicații multivoce definite pe spații metrice compacte (și care satisfac inegalitățile contractive (3) și (4) și inegalitățile expansive (5) și (6)) plecând de la rezultate asemănătoare obținute de B. Fisher [1], R. K. Jain și S. P. Dixit [3], Fazal-ur Rehman și B. Ahmad [2], T. Kubiak [4], T. Taniguchi [12], Nicoleta Negoescu [6]...[8], pentru alte tipuri de condiții contractive și expansive sau pentru operatori definiți pe spații de alt tip.

INVARIANCE FOR LINEAR DIFFERENTIAL EQUATIONS IN FINITE DIMENSIONAL ALGEBRAS

BY

CORNELIU URSESCU

Abstract. Invariant results concerning nonlinear differential equations in finite dimensional spaces are "translated" to linear differential equations in finite dimensional algebras.

Key words: differential equations, finite dimensional space, finite differential algebra, invariance.

1. Introduction

In this Note we take advantage of some results concerning invariance for nonlinear differential equations in finite dimensional spaces and we derive some results concerning invariance for linear differential equations in finite dimensional algebras (see § 2). Applications are given in case of norm inequalities (see § 3) and in case of component inequalities (see § 4). Final remarks (see § 5) concern the setting of generalized norms which subsumes both norm and component inequalities.

To begin with we recall the elementary invariance theory. Let X be a finite dimensional space, let $f : \mathbb{R} \times X \rightarrow X$ be a Carathéodory function, and consider the differential equation

$$\xi'(t) = f(t, \xi(t)).$$

Suppose, for every (τ, x) , there exists a unique saturated solution ξ to this differential equation such that $\xi(\tau) = x$ and denote this solution by $F(\cdot; \tau, x)$. The function F is said to be the evolutor generated by the differential equation above. It is continuous on its domain, which is open in $\mathbb{R} \times \mathbb{R} \times X$. Let P be a nonempty subset of X . The invariance problem is to find conditions under which $F(t; \tau, x) \in P$ for all $(t; \tau, x)$, with $t \geq \tau$ and $x \in P$. For differential equations with continuous right hand sides, such a condition can be derived from [15, p. 552]. For differential equations with Carathéodory right hand sides, a similar condition can be derived from [24, p. 484] (see also [23, p. 2] and the attribution in [16, p. 203], [17, p. 340], [18, p. 98]). The conditions are expressed in terms of the Bouligand-Severi tangency

concept ([3, p. 32], [19, p. 99]). We denote this tangency concept by K and the corresponding K -tangent set to P at a point x by $K_P(x)$.

Theorem 1. *Let the set P be closed. The following conditions are equivalent:*

- i) $F(t; \tau, x) \in P$ for all $(t; \tau, x)$, with $x \in P$ and $t \geq \tau$;
- ii) $f(t, x) \in K_P(x)$ for all x , with $x \in P$, and for almost all t .

Theorem 1 makes it possible to compare the solutions of two differential equations. Let Y be a finite dimensional space, let $g : \mathbb{R} \times Y \rightarrow Y$ be a Carathéodory function, and consider the differential equation

$$\eta'(t) = g(t, \eta(t)).$$

Suppose, for every (τ, y) , there exists a unique saturated solution η to this differential equation such that $\eta(\tau) = y$ and denote this solution by $G(\cdot; \tau, y)$. The function G is the evolver generated by the differential equation above. Let $P : X \rightarrow Y$ be a multifunction with nonempty graph. The comparison problem is to find conditions under which $G(t; \tau, y) \in P(F(t; \tau, x))$ for all $(t; \tau, x)$ and $(t; \tau, y)$, with $t \geq \tau$ and $y \in P(x)$. Such a condition is expressed in terms of the multifunction tangency concept generated by the set tangency concept K . The K -tangent multifunction to P at a point (x, y) is defined (cf. [1, p. 235]) through the equality

$$\text{graph}(K_P(x, y)) = K_{\text{graph}(P)}(x, y).$$

Applying Theorem 1 to the set $\text{graph}(P)$ we get Corollary 1 below.

Corollary 1. *Let $\text{graph}(P)$ be closed. The following two conditions are equivalent:*

- i) $G(t; \tau, y) \in P(F(t; \tau, x))$ for all $(t; \tau, x)$ and $(t; \tau, y)$, with $y \in P(x)$ and $t \geq \tau$;
- ii) $g(t, y) \in K_P(x, y)(f(t, x))$ for all (x, y) with $y \in P(x)$, and for almost all t .

2. Main Results

In this section we recast the preceding results in the algebra setting. Let X be a finite dimensional algebra, let $a : \mathbb{R} \rightarrow X$ be a locally integrable function, and consider the linear differential system:

$$A'(t, \tau) = a(t) * A(t, \tau),$$

$$A(\tau, \tau) = 1_X,$$

where 1_X stands for the unit of X . The function A is said to be the Cauchy evolver generated by the function a through the linear differential system above. Let P be a nonempty subset of X . The invariance problem is to find conditions under which $A(t, \tau) \in P$ for all (t, τ) , with $t \geq \tau$. To this end we have to suppose $1_X \in P$. The

condition is expressed in terms of the K -tangent set to P at 1_X , which we denote by Q .

Theorem 2. *Let $1_X \in P$, let the set P be closed, and let $P * P \subseteq P$. The following two conditions are equivalent:*

- i) $A(t, \tau) \in P$ for all (t, τ) , with $t \geq \tau$;
- ii) $a(t) \in Q$ for almost all t .

Proof. Denote $f(t, x) = a(t) * x$ and observe $F(t; \tau, x) = A(t, \tau) * x$. Because of the inclusion $P * P \subseteq P$, the former conditions in Theorems 1 and 2 are equivalent to each other. Because of the same inclusion, $Q * x \subseteq K_P(x)$ for all x , with $x \in P$, so that the latter conditions in Theorems 1 and 2 are also equivalent to each other. Now the conclusion follows.

Theorem 2 makes it possible to compare the solutions of two linear differential systems. Let Y be a finite dimensional algebra, let $b : \mathbb{R} \rightarrow Y$ be a locally integrable function, and consider the linear differential system

$$B'(t, \tau) = b(t) * B(t, \tau),$$

$$B(\tau, \tau) = 1_Y,$$

where 1_Y stands for the unit of Y . The function B is the Cauchy evolver generated by the function b through the linear differential system above. Let $P : X \rightarrow Y$ be a multifunction with nonempty graph. The comparison problem is to find conditions under which $B(t, \tau) \in P(A(t, \tau))$ for all (t, τ) , with $t \geq \tau$. To this end we have to suppose $1_Y \in P(1_X)$. The condition is expressed in terms of the K -tangent multifunction to P at $(1_X, 1_Y)$, which we denote by Q . Applying Theorem 2 to the set graph (P) , we get Corollary 2 below.

Corollary 2. *Let $1_Y \in P(1_X)$, let $\text{graph}(P)$ be closed, and let $P(x) * P(u) \subseteq P(x * u)$ for all x and u in $\text{domain}(P)$. The following conditions are equivalent:*

- i) $B(t, \tau) \in P(A(t, \tau))$ for all (t, τ) , with $t \geq \tau$;
- ii) $b(t) \in Q(a(t))$ for almost all t .

A final corollary concerns a slightly more complex setting. Consider also the locally integrable functions $c : \mathbb{R} \rightarrow X$ and $d : \mathbb{R} \rightarrow Y$ as well as the corresponding Cauchy evolvers $C : \mathbb{R} \times \mathbb{R} \rightarrow X$ and $D : \mathbb{R} \times \mathbb{R} \rightarrow Y$.

Corollary 3. *Let $1_Y \in P(1_X)$, let $\text{graph}(P)$ be a closed, convex cone, and let $P(x) * P(u) \subseteq P(x * u)$ for all x and u in $\text{domain}(P)$. The following conditions are equivalent:*

- i) $B(t, \tau) - D(t, \tau) \in P(A(t, \tau) - C(t, \tau))$ and $B(t, \tau) + D(t, \tau) \in P(A(t, \tau) + C(t, \tau))$ for all (t, τ) , with $t \geq \tau$;
- ii) $b(t) - d(t) \in P(a(t) - c(t))$ and $b(t) + d(t) \in Q(a(t) + c(t))$ for almost all t .

P r o o f. Let $X \times Y \times X \times Y$ be endowed with the product algebra structure and denote by E the set of all (x, y, u, v) such that

$$y - v \in P(x - u) \quad \text{and} \quad y + v \in P(x + u).$$

Then $(1_X, 1_Y, 1_X, 1_Y) \in E$. Further, $E * E \subseteq E$, i.e., $(x * \underline{x}, y * \underline{y}, u * \underline{u}, v * \underline{v}) \in E$ whenever $(x, y, u, v) \in E$ and $(\underline{x}, \underline{y}, \underline{u}, \underline{v}) \in E$, because of the identities:

$$2(z * \underline{z} - w * \underline{w}) = (z + w) * (\underline{z} - \underline{w}) + (z - w) * (\underline{z} + \underline{w});$$

$$2(z * \underline{z} + w * \underline{w}) = (z + w) * (\underline{z} + \underline{w}) + (z - w) * (\underline{z} - \underline{w}).$$

Finally, denote by F the K -tangent set to E at $(1_X, 1_Y, 1_X, 1_Y)$. Then F equals the set of all (x, y, u, v) such that

$$y - v \in P(x - u) \quad \text{and} \quad y + v \in Q(x + u).$$

Applying Theorem 2 in the appropriate case, we get the conclusion.

3. Norm Inequalities

Throughout this section $Y = R$, ν is a norm on X , and λ is the corresponding logarithmic norm on X (see [9, p. 58], [4]);

$$\lambda(x) = \lim_{h \rightarrow 0^+} \frac{1}{h} (\nu(1_X + hx) - \nu(1_X)).$$

Further, suppose $\nu(x * u) \leq \nu(x)\nu(u)$ for all x and u (see [14, p. 62]) as well as $\nu(1_X) = 1$ (see [5, p. 4]). Next, we paraphrase Corollaries 2 and 3 in case $P(x) = \{y; \nu(x) \leq y\}$. In this case, $Q(x) = \{y; \lambda(x) \leq y\}$. Proposition 1 below enriches a result in [9, p. 61] concerning the estimation of $\nu(A(t; \tau))$ through $\lambda(a(t))$.

P r o p o s i t i o n 1. For all (t, τ) , with $t \geq \tau$,

$$\nu(A(t, \tau)) \leq \int_{\tau}^t \lambda(a(\theta)) d\theta,$$

and this estimation is the best one.

P r o o f. Apply Corollary 2 and take $b(t) = \lambda(a(t))$.

Proposition 2 below, which reduces to Proposition 1 in case $a = c$, provides a quantitative refinement of some inequalities from [11, p. 1267] concerning the estimations of $\nu(A(t, \tau) - B(t, \tau))$ and $\nu(A(t, \tau) + B(t, \tau))$ through $\lambda(a(t) + b(t))$ and $\nu(a(t) - b(t))$.

Proposition 2. For all (t, τ) , with $t \geq \tau$,

$$\begin{aligned} & \frac{1}{2} \nu(A(t, \tau) - C(t, \tau)) \leq \\ & \leq \exp \left[\int_{\tau}^t \frac{1}{2} \lambda(a(\theta) + c(\theta)) d\theta \right] \sinh \left[\int_{\tau}^t \frac{1}{2} \nu(a(\theta) - c(\theta)) d\theta \right], \\ & \frac{1}{2} \nu(A(t, \tau) + C(t, \tau)) \leq \\ & \leq \exp \left[\int_{\tau}^t \frac{1}{2} \lambda(a(\theta) + c(\theta)) d\theta \right] \cosh \left[\int_{\tau}^t \frac{1}{2} \nu(a(\theta) - c(\theta)) d\theta \right], \end{aligned}$$

and these estimations are the best ones.

Proof. Apply Corollary 3 and take $b(t) - d(t) = \nu(a(t) - c(t))$ as well as $b(t) + d(t) = \nu(a(t) + c(t))$.

4. Component Inequalities

First, let X be the algebra of real, square n -matrices. Next, we paraphrase Corollary 1 in case P is the set of all x such that $0 \leq x_{ij}$ for all i and j . In this case, Q is the set of all x such that $0 \leq x_{ij}$ for all i and j , with $i \neq j$. Proposition 3 below copies out a result in [7, p. 54] (cf. [2, p. 376]).

Proposition 3. The following conditions are equivalent:

- i) $0 \leq A_{ij}(t, \tau)$ for all i and j , and for all (t, τ) , with $t \geq \tau$;
- ii) $0 \leq a_{ij}(t)$ for all i and j , with $i \neq j$, and for almost all.

More general, nonlinear results of this type may be found in [13, p. 642], [6, p. 82], [26, p.143], and [21, p. 56].

Finally, let X be the algebra of complex, square n -matrices and let Y be the algebra of real, square n -matrices. Next we paraphrase Corollary 2 in case $P(x)$ is the set of all y such that $|x_{ij}| \leq y_{ij}$ for all i and j . In this case, $Q(x)$ is the set of all y such that $\Re(x_{ii}) \leq y_{ii}$ and $|x_{ij}| \leq y_{ij}$ for all i and j with $i \neq j$. Here $\Re(c)$ stands for the real part of the complex number c . Such matrix relations have been initially used and afterwards investigated in [8, p. 226], [10, p. 809]. Proposition 4 below copies out a result in [11, p. 1269].

Proposition 4. The following conditions are equivalent:

- i) $|A_{ij}(t, \tau)| \leq B_{ij}(t, \tau)$ for all i and j , and for all (t, τ) , with $t \geq \tau$;
- ii) $\Re(a_{ii}(t)) \leq b_{ii}(t)$ and $|a_{ij}(t)| \leq b_{ij}(t)$ for all i and j , with $i \neq j$, and for almost every t .

More general, nonlinear results of this type may be found in [8, p. 226] and [25, p. 90]).

5. Final Remarks

Both Propositions 2 and 4 have been extended in [12] to a pair of abstract algebras X and Y . There, Y is endowed with an order $\leq$, whereas X is endowed with a generalized norm $\nu : X \rightarrow Y$. That theory can be retrieved from our theory just because the abundance of properties imposed there upon the pair $\leq$ and ν result in the few properties imposed here upon P in case $P(x) = \{y; \nu(x) \leq y\}$. Furthermore, $Q(x) = \{y; \lambda(x) \leq y\}$ whenever the generalized logarithmic norm λ is defined by the equality

$$\lambda(x) = \lim_{\substack{s \rightarrow 0+ \\ u \rightarrow 0}} \frac{1}{s} (\nu(1_X + s(x+u)) - \nu(1_X)),$$

which presumes ν is Severi differentiable [20, p. 20] at 1_X and which means λ is the Severi differential [22, p. 200] of ν at 1_X . Standard examples are provided by § 3 and § 4 above. In the first case $Y = \mathbb{R}$ and ν is a norm on X . In the second case X is the algebra of complex, square n -matrices, Y is the algebra of real, square n -matrices, $\nu(x)$ is the matrix of which elements are $|x_{ij}|$ and $\lambda(x)$ is the matrix of which diagonal elements are $\Re(x_{ii})$ and of which off-diagonal elements are $|x_{ij}|$. In the general case, paraphrases of Corollaries 2 and 3 are also available.

Proposition 5. *The following two conditions are equivalent:*

i) $\nu(A(t, \tau)) \leq B(t, \tau)$ for all (t, τ) , with $t \geq \tau$;

ii) $\lambda(a(t)) \leq b(t)$ for almost every t .

Proposition 6. *The following two conditions are equivalent:*

i) $\nu(A(t, \tau) + C(t, \tau)) \leq B(t, \tau) + D(t, \tau)$ and $\nu(A(t, \tau) - C(t, \tau)) \leq B(t, \tau) - D(t, \tau)$ for all (t, τ) with $t \geq \tau$;

ii) $\lambda(a(t) - c(t)) \leq b(t) + d(t)$ and $\nu(a(t) - c(t)) \leq b(t) - d(t)$ for almost all t .

Since $\leq$ is reflexive, the best estimations occur when either $b(t) = \lambda(a(t))$, or $b(t) + d(t) = \lambda(a(t) + c(t))$ and $b(t) - d(t) = \nu(a(t) - c(t))$.

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REFERENCES

1. Aubin J. P., *Comportement lipschitzien des solutions de problèmes de minimization convexes*. C. R. Acad. Sci., Sér. I Math., **295**, 3, 235-238 (1982).
2. Bellman R., Glicksberg I., Gross O., *On Some Variational Problems Occuring in the Theory of Dynamic Programming*. Rend. Circ. Mat. Palermo (2), **3**, 363-397 (1954).
3. Bouligand G., *Sur les surfaces dépourvues de points hyperlimites (ou: un Théorème d'existence du plan tangent.)* Ann. Soc. Polon. Math., **9**, 32-41 (1930).

4. Dahlquist G., *Stability and Error Bounds in the Numerical Integration of Ordinary Differential Equations*. Kungl. Tekn. Högsk. Handl. Stockholm, 130 (1959).
5. Gelfand I., *Normierte Ringe*. Mat. Sb., 9 (51), 3-24 (1941).
6. Kamke E., *Zur Theorie der Systeme gewöhnlicher Differentialgleichungen (II)*. Acta Math., 58, 57-85 (1932).
7. Loewner C., *A Theorem on the Partial Order Derived from a Certain Transformation Semigroup*. Math. Z., 72, 53-60 (1959).
8. Lozinskii S. M., *Estimate of the Error of an Approximate Solution of a System of Ordinary Differential Equations* (in Russian). Dokl. Akad. Nauk SSSR, 92, 2, 225-228 (1953).
9. Lozinskii S. M., *Error Estimate for Numerical Integration of Ordinary Differential Equations (I)* (in Russian). Izv. Vyssh. Uchebn. Zaved. Mat., 5(6), 52-90 (1959); errata, 5(12), 222 (1959).
10. Lozinskii S. M., *On the Theory of Finite Matrices* (in Russian). Dokl. Akad. Nauk SSSR, 163, 4, 809-811 (1965).
11. Lozinskii S. M., *Estimation of the Difference of Two Matrizants* (in Russian). Dokl. Akad. Nauk SSSR, 172, 6, 1266-1269 (1967).
12. Lozinskii S. M., *Matrix Norm and Their Applications to the Estimation of the Eigenvalues of Matrices and to Differential Equations* (in Russian). Vestnik Leningrad. Univ. Mat. Mekh. Astronom., 23, 13, 51-61 (1968).
13. Müller W., *Über das Fundamentaltheorem in der Theorie der gewöhnlicher Differentialgleichungen*. Math. Z., 26, 619-645 (1927).
14. Nagumo M., *Einige analytische Untersuchungen in linearen, metrischen Ringen*. Japon. J. Math., 13, 61-80 (1936).
15. Nagumo M., *Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen*. Proc. Phys.-Math. Soc. Japan (3), 24, 551-559 (1942).
16. Pavel N., *Differential Equations Associated to Some Nonlinear Operators on Banach Spaces* (in Romanian). Ed. Academiei R.S.R., București, 1977.
17. Pavel N. H., *Analysis of Some Nonlinear Problems in Banach Spaces and Applications*. Univ. Al. I. Cuza", Iași, Facultatea de Matematică, 1982.
18. Pavel N. H., *Differential Equations, Flow Invariance and Applications*. Pitman, Boston, 1984.
19. Severi F., *Su alcune questioni di topologia infinitesimale*. Ann. Soc. Polon. Math., 9, 97-108 (1930).
20. Severi F., *Sulla differenziabilità totale delle funzioni di più variabili reali*. Ann. Mat. Pura Appl. (4), 13, 1-35 (1935).
21. Szarski J., *Differential Inequalities*. P.W.N., Warsaw, 1967.
22. Ursescu C., *Sur une généralisation de la notion de différentiabilité*. Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur. (7), 54, 199-204 (1973).
23. Ursescu C., *Carathéodory Solutions of Ordinary Differential Equations on Locally Compact Sets in Fréchet Spaces*. Preprint Series in Mathematics of "Al. Myller" Mathematical Seminary, No. 18, University "Al. I. Cuza" Jassy, 1982.
24. Ursescu C., *Carathéodory Solutions of Ordinary Differential Equations on Locally Closed Sets in Finite Dimensional Spaces*. Math. Japon., 31, 483-491 (1986).
25. Walter W., *Differential- und Integral-Ungleichungen*. Springer, Berlin, 1964.
26. Wazewski T., *Systèmes des équations et des inégalités différentielles ordinaires aux deuxièmes membres monotones et leurs applications*. Ann. Soc. Polon. Math., 23, 112-166 (1950).

INVARIANTĂ PENTRU ECUAȚII DIFERENȚIALE LINIARE ÎN ALGEBRE FINIT DIMENSIONALE

(Rezumat)

Rezultate de invariantă din teoria ecuațiilor diferențiale neliniare în spații finit dimensionale sunt transpuse pentru ecuații diferențiale liniare în algebre finit dimensionale.

ON L^P -STABILITY IN VARIATION OF FUNCTIONAL DIFFERENTIAL EQUATIONS

BY

PAVEL TALPALARU

Abstract. In this paper are established relations between different kinds of stability in variation of the zero solution of Eq. (2.1) and classical types of stability of the zero solution of perturbed Eq. (2.2).

Key words: ordinary differential equation, stability, asymptotic behaviour, growth properties.

1. Introduction

The study of stability and asymptotic behaviour of solutions of ordinary differential equations has made considerable use of the variation of constants formula, both the classical linear version and the nonlinear version of Aleksiev [1]. In [5] G. A. Shanholdt has developed an analogue of Alekseev's formula for use with perturbed nonlinear functional differential equation. In this paper, the Shanholdt's representation will be used to obtain some results concerning the so-called stability in variation and growth properties for functional differential equations.

2. Preliminaries

In the following we will use the standard notation introduced by J. K. Hale [2]. Suppose the real numbers r and τ are fixed with $r \geq 0$ and $\tau \in \mathbb{R}$, E^n is an n -dimensional linear vector space with norm $|\cdot|$ and $C([a, b], E^n)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into E^n with the topology of uniform convergence. When $[a, b] = [-r, 0]$, we let $C = C([-r, 0], E^n)$, and denote the norm of $\varphi \in C$ by $\|\varphi\| = \sup \{|\varphi(\theta)|; -r \leq \theta \leq 0\}$. If $x \in C([a - r, b], E^n)$ for any $a \leq b$, then for each fixed $t \in [a, b]$, the symbol x_t will denote an element of the space C defined by $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$.

Let Λ be an open subset of C , $\Gamma = (\tau, \infty) \times \Lambda$, and $f : \Gamma \rightarrow E^n$ be a given

function. A functional differential equation is relation of the form

$$(2.1) \quad \dot{x}(t) = f(t, x_t),$$

where $\dot{x}(t)$ denotes the right-hand derivative of $x(u)$ at $u = t$.

For any $\sigma \in (\tau, \infty)$, $\varphi \in C$, we say that $x = x(\sigma, \varphi)$ is a solution of (2.1) with initial function φ at σ or a solution of (2.1) through (σ, φ) if there exists an $a > 0$ such that $x \in C([\sigma - r, \sigma + a], E^n)$, $x_\sigma = \varphi$ and x satisfies (2.1) for $t \in [\sigma, \sigma + a]$.

We are interested in establishing relationships between the solutions of (2.1) and those of perturbed equations

$$(2.2) \quad \dot{y}(t) = f(t, y_t) + g(t, y_t).$$

It is assumed that $f, g : \Gamma \rightarrow E^n$ are continuous in Γ , and that $f(t, \varphi)$ has a continuous Fréchet derivative $f'(t, \varphi)$ with respect to φ in Γ .

Corresponding to each solution $x(\sigma, \varphi)$ of (2.1), one defines a linear functional differential equation

$$(2.3) \quad \dot{z} = f'(t, x_t(\sigma, \varphi)) z_t, \quad t \in J(\sigma, \varphi),$$

where $J(\sigma, \varphi)$ is the maximal interval of existence of $x_t(\sigma, \varphi)$.

Eq. (2.3) is called the linear variational equation of (2.1) with respect to $x(\sigma, \varphi)$. In what follows, the family of linear operators associated with (2.3) will be denoted by $T(t, \sigma : \varphi)$, $t \geq \sigma$.

If the $n \times n$ matrix function Y_0 is defined by: $Y_0(\theta) = 0$, $-r \leq \theta < 0$, $Y(0) = I$ - the unity matrix, then (Theorem 3, [5]), for any $(\sigma, \varphi) \in (\tau, \infty) \times \Lambda$, if $J(\sigma, \varphi) = [\sigma, \infty)$ and solutions of (2.2) are unique, we have

$$(2.4) \quad y_t(\sigma, \varphi) = x_t(\sigma, \varphi) + \int_{\sigma}^t T(t, s : y_s(\sigma, \varphi)) Y_0 g(t, s, y_s(\sigma, \varphi)) ds,$$

so long as $(\sigma, \varphi) \in (\tau, \infty) \times \Lambda$ and t is in an interval in which $y_t(\sigma, \varphi)$ exists.

We also note that (Theorem 2, [5]) if $\Lambda_0 \subset \Lambda$ is a convex set and for every $(\sigma, \varphi) \in (\tau, \infty) \times \Lambda_0$, $J_0(\sigma, \varphi) \equiv [\sigma, \infty)$, then

$$(2.5) \quad x_t(\sigma, \varphi_2) - x_t(\sigma, \varphi_1) = \int_0^1 T(t, \sigma : \varphi_1 + \xi(\varphi_2 - \varphi_1)) d\xi(\varphi_2 - \varphi_1),$$

provided $(\sigma, \varphi_1), (\sigma, \varphi_2) \in (\tau, \infty) \times \Lambda_0$, $0 \leq \xi \leq 1$.

A noteworthy particular case of (2.5) is obtained for $f(t, 0) \equiv 0$, namely

$$(2.6) \quad x_t(\sigma, \varphi) = \int_0^1 T(t, \sigma : \xi \varphi) d\xi \varphi.$$

We now give the definitions of different kinds of stability in terms of the behaviour of solutions of (2.1) and of the variational Eq. (2.3). In what follows we shall assume that $f(t, 0) = g(t, 0) = 0$.

Definition. The solution $x = 0$ of (2.1) is said to be:

a) *Globally asymptotically stable* (GAS) if it is asymptotically stable and all its solutions tend to zero;

b) *Uniformly L^P -stable* (UL^P -S), $p \geq 1$, if it is uniformly stable (US) and

$$\int_{\sigma}^{\infty} \|x_t(\sigma, \varphi)\|^p dt < \infty \quad \text{for all } \varphi \in C, \quad \|\varphi\| < h \leq \infty;$$

c) *Generalized exponentially asymptotically stable in variation* (GEASV) if there exist a constant $N > 0$ and a positive strictly increasing function $\alpha(t)$ defined on $[0, \infty)$, $\alpha(0) = 0$, $\alpha(t) \rightarrow \infty$, such that $\|T(t, \sigma; \varphi)\| \leq N \exp[\alpha(\sigma) - \alpha(t)]$ for all $t \geq \sigma$, $\varphi \in C$, $\|\varphi\| < H \leq \infty$; If $\alpha(t) = vt$, $v = \text{const.} > 0$, then $x = 0$ is *exponentially asymptotically stable in variation* (EASV);

d) *Uniformly stable in variation* (USV) if there exists $M > 0$ such that $\|T(t, \sigma; \varphi)\| \leq M$ for all $t \geq \sigma$ and $\varphi \in C$, $\|\varphi\| < H \leq \infty$;

e) *Uniformly L^P -stable in variation* (UL^P -SV) if it is (USV) and for any solution $x_t(\sigma, \varphi)$,

$$\varphi \in C, \quad \|\varphi\| < H \leq \infty, \quad \text{we have} \quad \int_{\sigma}^{\infty} \|x_t(\sigma, \varphi)\|^p dt < \infty.$$

3. Main Results

In this part we shall establish relationships between different kinds of stability in variation of the zero solution of Eq. (2.1) and classical types of stability of the zero solution of perturbed Eq. (2.2).

Theorem 3.1. Assume that the solution $x = 0$ of (2.1) is (UL^1 -SV) and that

$$(3.1) \quad |g(t, y_t)| \leq \lambda(t) \|y_t\| \quad \text{for } t \geq \sigma \text{ and } y_t \in C, \quad \|y_t\| < H \leq \infty,$$

where $\lambda(t)$ is a nondecreasing positive function such that $\lambda(t) \in UL^1([\sigma, \infty))$. Then the zero solution of (2.2) is (UL^1 -S).

Proof. We shall show first that the solution $y = 0$ of (3.2) is (US). Indeed, for any

$$0 < \varepsilon < H, \quad \text{let } \delta(\varepsilon) < \frac{\varepsilon}{M \exp(KM)} \quad \text{and } \|\varphi\| < \delta, \quad \text{where } K = \int_{\sigma}^{\infty} \lambda(t) dt.$$

Suppose, by contrary, that there exists $t_1 \geq \sigma$ such that $\|y_{t_1}(\sigma, \varphi)\| = \varepsilon$ and $\|y_t(\sigma, \varphi)\| < \varepsilon$ on $[\sigma, t_1)$. By variation of constants formula (2.4) and by using (2.6) in our hypothesis that $x = 0$ is (USV) and (3.1) we have

$$(3.2) \quad \|y_t(\sigma, \varphi)\| \leq M\|\varphi\| + M \int_{\sigma}^t \lambda(s) \|y_s(\sigma, \varphi)\| ds \quad \text{on } [\sigma, t_1).$$

From here, by applying the Gronwall inequality, one obtains

$$\|y_t(\sigma, \varphi)\| \leq M\|\varphi\| \exp \left(M \int_{\sigma}^t \lambda(s) ds \right) \leq M \exp(MK) \|\varphi\| \quad \text{on } [\sigma, t_1).$$

Therefore, $\|y_{t_1}(\sigma, \varphi)\| < \varepsilon$ which is a contradiction and thus the zero solution of (2.2) is (US). To show that $\int_{\sigma}^{\infty} \|y_t(\sigma, \varphi)\| dt < \infty$ whenever $\varphi \in C$, $\|\varphi\| < H \leq \infty$, we shall use (2.4) from where, for an arbitrary $u \geq \sigma$, we have

$$\int_{\sigma}^u \|y_t(\sigma, \varphi)\| dt \leq \int_{\sigma}^t \|x_t(\sigma, \varphi)\| dt +$$

$$+ M \int_{\sigma}^u \left(\int_{\sigma}^t \lambda_s \|y_s(\sigma, \varphi)\| ds \right) dt \leq K_1 + M \int_{\sigma}^u \lambda(t) \left(\int_{\sigma}^t \|y_s(\sigma, \varphi)\| ds \right) dt,$$

where $K_1 = \int_{\sigma}^{\infty} \|x_t(\sigma, \varphi)\| dt$. Again, by the Gronwall inequality we get

$$\int_{\sigma}^u \|y_t(\sigma, \varphi)\| dt \leq K_1 \exp \left(M \int_{\sigma}^u \lambda(t) dt \right) \leq K_1 \exp(MK),$$

for any $u \geq \sigma$, therefore

$$\int_{\sigma}^{\infty} \|y_t(\sigma, \varphi)\| dt < \infty \quad \text{for any solution } y_t(\sigma, \varphi), \quad \varphi \in C, \quad \|\varphi\| < H \leq \infty.$$

Consequently, the zero solution of (2.2) is (UL¹ - S).

C o r o l l a r y 3.1. Assume that the solution $x = 0$ of (2.1) is (GEASV) and that (3.1) holds with $\lambda(t)$ a positive function such that $\lambda \in L^1([\sigma, \infty))$. Then the zero solution of (2.2) is (GAS).

P r o o f. By closed arguments used in the proof of Theorem 3.1 we have

$$\|y_t(\sigma, \varphi)\| \leq N\|\varphi\| + N \int_{\sigma}^t \exp[\alpha(s) - \alpha(t)] \lambda(s) \|y_s(\sigma, \varphi)\| ds \leq$$

$$\leq N\|\varphi\| + N \int_{\sigma}^t \lambda(s) \|y_s(\sigma, \varphi)\| ds,$$

from where $\|y_t(\sigma, \varphi)\| \leq NK\|\varphi\|$. To show that all solutions of (2.2) tend to zero as $t \rightarrow \infty$, we shall use (2.4), (2.6) and the (GEASV) of the solution $x = 0$. One obtains

$$\begin{aligned} \|y_t(\sigma, \varphi)\| \exp \alpha(t) &\leq N\|\varphi\| \exp \alpha(\sigma) \exp \left(N \int_{\sigma}^t \lambda(s) ds \right) \leq \\ &\leq N \exp(NK) \exp \alpha(\sigma) \|\varphi\|, \end{aligned}$$

from where, finally, we have

$$\|y_t(\sigma, \varphi)\| \leq N \exp(NK) \|\varphi\| \exp[\alpha(\sigma) - \alpha(t)], \quad t \geq \sigma, \quad \varphi \in C, \quad \|\varphi\| < H \leq \infty.$$

This last inequality yields to desired result.

Corollary 3.2. Assume that the solution $x = 0$ of (2.1) is (EASV) and (3.1) holds with $\lambda(t)$ as in Corollary 3.1. Then the zero solution of (2.2) is (UL¹-S) and, in addition, for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\int_{\sigma}^{\infty} \|y_t(\sigma, \varphi)\| dt < \varepsilon \quad \text{for any } \varphi \in C, \quad \|\varphi\| < \delta.$$

Proof. As above one obtains

$$\begin{aligned} \|y_t(\sigma, \varphi)\| &\leq N\|\varphi\| \exp(-\nu t + \nu\sigma) \exp \left(N \int_{\sigma}^t \lambda(s) ds \right) \leq \\ &\leq N\|\varphi\| \exp(-\nu t + \nu\sigma) \exp(NK) \leq N \exp(NK) \|\varphi\|, \end{aligned}$$

from where it follows the (US¹-S) of the zero solution of (2.2). Moreover,

$$\int_{\sigma}^{\infty} \|y_t(\sigma, \varphi)\| dt < \frac{N}{\nu} \|\varphi\| \exp(NK) < \varepsilon \quad \text{if } \|\varphi\| < \frac{\varepsilon \nu}{N} \exp(-NK) = \delta(\varepsilon).$$

Remark 3.1. Corollary 3.2 is an extension to the perturbed Eq. (2.2) of Theorem 2 [5].

Corollary 3.3. Assume that the solution $x = 0$ of (2.1) is (UL¹-SV) and that

$$\begin{aligned} (3.3) \quad |g(t, y_t)| &\leq \lambda(t) \|y_t\|^m, \quad 0 < m < 1 \quad \text{for all } t \geq \sigma \\ &\text{and } y_t \in C, \quad \|y_t\| < H \leq \infty. \end{aligned}$$

Furthermore, we assume that

$$v(t) \equiv \left(\int_{\sigma}^t \lambda(s)^r ds \right)^{1/r} \in L^1([\sigma, \infty)), \quad r = \frac{1}{1-m}.$$

Then all solutions of (2.2) belong to $L^1([\sigma, \infty))$.

P r o o f. By similar arguments used in the proof of Theorem 3.1 and by Hölder's inequality we obtain

$$\begin{aligned} \|y_t(\sigma, \varphi)\| &\leq \|x_t(\sigma, \varphi)\| + M \int_{\sigma}^t \lambda(s) \|y_t(\sigma, \varphi)\|^m ds \leq \\ &\leq \|x_t(\sigma, \varphi)\| + M \left(\int_{\sigma}^t \lambda(s)^r ds \right)^{1/r} \left(\int_{\sigma}^t \|y_s(\sigma, \varphi)\|^m ds \right)^{m/r}. \end{aligned}$$

From here, for arbitrary $u \geq \sigma$ we get

$$\int_{\sigma}^u \|y_t(\sigma, \varphi)\| dt \leq K_1 + M \int_{\sigma}^u v(t) \left(\int_{\sigma}^t \|y_s(\sigma, \varphi)\|^m ds \right)^{m/r} dt, \quad K_1 = \int_{\sigma}^{\infty} \|x_t(\sigma, \varphi)\| dt < \infty.$$

Using the Gronwall inequality [3, Theorem 1.3.4, Ex. 1.3.1, p.21] it follows

$$\int_{\sigma}^u \|y_t(\sigma, \varphi)\| dt \leq \left(K_1^{1-m} + (1-m)M \int_{\sigma}^u v(t) dt \right),$$

for any $u \geq \sigma$, and hence $y_t(\sigma, \varphi) \in L^1([\sigma, \infty))$. On the other hand, if we take into account (2.6), one obtains

$$\|y_t(\sigma, \varphi)\| \leq M\|\varphi\| + M \int_{\sigma}^t \lambda(s) \|y_s(\sigma, \varphi)\|^m ds, \quad t \geq \sigma,$$

from where the estimation

$$\|y_t(\sigma, \varphi)\| \leq \left((M\|\varphi\|)^{1-m} + (1-m)M \int_{\sigma}^t \lambda(s) ds \right), \quad t \geq \sigma, \quad \varphi \in C.$$

Theorem 3.2. Assume that the solution $x = 0$ of (2.1) is $(UL^p - SV)$ and that (3.1) holds with $\lambda(t) \in L^1([\sigma, \infty))$ and, in addition,

$$v(t) \equiv \left(\int_{\sigma}^t \lambda(s)^q ds \right)^{p/q} \in L^1([\sigma, \infty)), \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Then the zero solution (2.2) is $(UL^p - S)$.

P r o o f. The solution $y = 0$ of (2.2) is (US) , according to the proof of Theorem 3.1. To show that

$$\int_{\sigma}^{\infty} \|y_t(\sigma, \varphi)\|^p dt < \infty \quad \text{whenever} \quad \varphi \in C, \quad \|\varphi\| < H \leq \infty,$$

we shall use again formula (2.4) and (3.1) to obtain

$$\begin{aligned} \|y_t(\sigma, \varphi)\|^p &\leq 2^{p-1} \|x_t(\sigma, \varphi)\|^p + 2^{p-1} M^p \left(\int_{\sigma}^t \lambda(s) \|y_s(\sigma, \varphi)\| ds \right)^p \leq \\ &\leq 2^{p-1} \|x_t(\sigma, \varphi)\|^p + 2^{p-1} M^p \left(\int_{\sigma}^t \lambda(s)^q ds \right)^{p/q} \int_{\sigma}^t \|y_s(\sigma, \varphi)\|^p ds. \end{aligned}$$

From here, for arbitrary $u \geq \sigma$, we get

$$\int_{\sigma}^u \|y_t(\sigma, \varphi)\|^p dt \leq K + 2^{p-1} M^p \int_{\sigma}^u v(t) \left(\int_{\sigma}^t \|y_s(\sigma, \varphi)\|^p ds \right) dt,$$

where

$$K = 2^{p-1} \int_{\sigma}^{\infty} \|x_t(\sigma, \varphi)\|^p dt < \infty.$$

By Gronwall's inequality it follows

$$\int_{\sigma}^u \|y_t(\sigma, \varphi)\|^p dt \leq K \exp \left(2^{p-1} M^p \int_{\sigma}^u v(t) dt \right) \leq K \exp (2^{p-1} M^p L),$$

for any $u \geq \sigma$, where $L = \int_{\sigma}^{\infty} v(t) dt < \infty$. Therefore the zero solution of (3.2) is (UL^p-S) .

Remark 3.2. If we assume that the solution $x = 0$ of (2.1) is (EASV) the Theorem 3.2 appears as an extension of Theorem 3 [5].

C o r o l l a r y 3.4. Assume that the solution $x = 0$ of (2.1) is (UL^p-SV) and that (3.3) holds. Furthermore, we assume that

$$v(t) \equiv \left(\int_{\sigma}^t \lambda(s)^r ds \right)^{1/r} \in L^1([\sigma, \infty)), \quad r = \frac{p}{p-m}.$$

Then all solutions of (2.2) belong to $L^p((\sigma, \infty))$.

P r o o f. We only note that, this time, we have

$$\|y_t(\sigma, \varphi)\|^p \leq 2^{p-1} \|x_t(\sigma, \varphi)\|^p + 2^{p-1} M^p \left(\int_{\sigma}^t \lambda(s)^r ds \right)^{1/r} \left(\int_{\sigma}^t \|y_s(\sigma, \varphi)\|^p ds \right)^m.$$

Further, a reasoning similar to one used to prove Theorem 3.2 will be handled.

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REFERENCES

1. Alekseev V. M., *An Estimate for the Perturbations of the Solutions of Ordinary Differential Equations* (in Russian). Vestnik Moskov. Univ., Ser. 1, Mat. Mech., 2, 28-36 (1961).
2. Hale J. K., *Functional Differential Equations*. Springer Verlag, New York, 1971.
3. Lakshmikantham V., Leela S., Martynyuk A. A., *Stability Analysis of Nonlinear Systems*. Marcel Dekker, Inc., New York and Basel, 1989.
4. Morchalo J., *Liapunov Functions and Asymptotic Behaviour of the Solutions of Differential-Difference Equations*. Studia Univ. "Babeș-Bolyai", Mathematica, XXXVII, 4, 43-55 (1992).
5. Shanholdt G. A., *A Nonlinear Variation- of-Constants Formula for Functional Differential Equations*. Math. Systems Theory, 6, 4, 343-352 (1973).

ASUPRA L^p -STABILITĂȚII ÎN VARIATII A
ECUAȚIILOR DIFERENȚIALE FUNCȚIONALE

(Rezumat)

Se stabilesc relații între diferite tipuri de stabilitate în variații a soluției zero a ecuației (2.1) și tipuri clasice de stabilitate a soluției zero a ecuației perturbate (2.2).

OBSERVATIONS ON A NONLINEAR HYPERBOLIC PROBLEM

BY

RODICA LUCA

Abstract. It is stated and proved a result on the existence and unicity for the integral solutions of the problem (S), (BC), (IC) in the space $C([0, 1]; (L^1(0, 1))^2)$. Some basic concepts and results in the theory of accretive operators and nonlinear equations of monotone type are involved.

Key words: hyperbolic systems, accretive operators, nonlinear equations of monotone type.

1. Introduction

In this paper we shall study the hyperbolic system:

$$(S) \quad \begin{cases} \frac{\partial i}{\partial t} + \frac{\partial v}{\partial x} + \alpha(i) = f(t, x), \\ \frac{\partial v}{\partial t} + \frac{\partial i}{\partial x} + \beta(v) = g(t, x), \end{cases}$$

$$0 < x < 1, \quad 0 < t < T,$$

with the boundary conditions

$$(BC) \quad i(t, 0) \in -f_1(v(t, 0)), \quad i(t, 1) \in f_2(v(t, 1)), \quad 0 < t < T$$

and the initial data

$$(IC) \quad i(0, x) = i_0(x), \quad v(0, x) = v_0(x), \quad 0 < x < 1.$$

We denote by (H.1) and (H.2) the following assumptions:

(H.1) The functions $\alpha, \beta : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz continuous.

(H.2) The operators $f_1 : D(f_1) \subset \mathbb{R} \rightarrow \mathbb{R}$ and $f_2 : D(f_2) \subset \mathbb{R} \rightarrow \mathbb{R}$ are maximal monotone (possibly multivalued) with $D(f_1) \neq \emptyset$, $D(f_2) \neq \emptyset$.

We shall establish an existence result for the solutions of the problem (S), (BC), (IC) in the space $C([0, T]; (L^1(0, 1))^2)$. This problem with more general boundary conditions has been investigated in Hilbert spaces in [2], [3]. For the basic concepts and results in the theory of accretive operators and nonlinear equations of monotone type we refer the reader to [1].

2. Results

We consider the space $X = (L^1(0, 1))^2$ with the norm

$$\left\| \begin{pmatrix} i \\ v \end{pmatrix} \right\|_X = \int_0^1 \max(|i(x)|, |v(x)|) dx$$

and define the operator $A : D(A) \subset X \rightarrow X$,

$$D(A) = \left\{ \begin{pmatrix} i \\ v \end{pmatrix} \in (W^{1,1}(0, 1))^2; i(0) \in -f_1(v(0)), i(1) \in f_2(v(1)) \right\},$$

$$A \begin{pmatrix} i \\ v \end{pmatrix} = \begin{pmatrix} v' \\ i' \end{pmatrix}.$$

Remark 1. Under the assumption (H.2), $D(A) \neq \emptyset$ and $\overline{D(A)} = X$. Indeed, because $D(f_1) \neq \emptyset$ and $D(f_2) \neq \emptyset$ there exist $a, b, c, d \in \mathbb{R}$ such that $b \in -f_1(a)$ and $d \in f_2(c)$. Then the functions $i(x) = (d-b)x + b$ and $v(x) = (c-a)x + a$ satisfy the conditions of $D(A)$. Moreover, if $\begin{pmatrix} i \\ v \end{pmatrix} \in D(A)$ then $\begin{pmatrix} i \\ v \end{pmatrix} + (C_0^\infty(0, 1))^2 \subset D(A)$. For, if $\begin{pmatrix} f \\ g \end{pmatrix} \in (C_0^\infty(0, 1))^2$ then we have $(i+f)(0) \in -f_1((v+g)(0))$ and $(i+f)(1) \in f_2((v+g)(1))$, so $\begin{pmatrix} i+f \\ v+g \end{pmatrix} \in D(A)$. Now, from $(C_0^\infty(0, 1))^2 = X$, we deduce that $\overline{D(A)} = X$.

Lemma 1. If (H.2) holds then the operator A is accretive in the space X .

Lemma 2. If (H.2) holds then the operator A is m -accretive in X .

Let us define the function $F : X \rightarrow X$, $F \begin{pmatrix} i \\ v \end{pmatrix} = \begin{pmatrix} \alpha(i) \\ \beta(v) \end{pmatrix}$, which is, evidently, under the assumption (H.1), Lipschitz continuous in X .

Now, using the operators A and F , our problem (S), (BC), (IC) can be equivalently expressed as a Cauchy problem in the space X :

$$(P) \quad \begin{cases} \frac{d}{dt} \begin{pmatrix} i(t) \\ v(t) \end{pmatrix} + A \begin{pmatrix} i(t) \\ v(t) \end{pmatrix} + F \begin{pmatrix} i(t) \\ v(t) \end{pmatrix} \ni \begin{pmatrix} f(t, \cdot) \\ g(t, \cdot) \end{pmatrix}, & 0 < t < T; \\ \begin{pmatrix} i(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} i_0 \\ v_0 \end{pmatrix}. \end{cases}$$

We say that $\begin{pmatrix} i \\ v \end{pmatrix}$ is a integral solution for the problem (S), (BC) if $\begin{pmatrix} i \\ v \end{pmatrix}$ is a integral solution for the equation (P)₁.

From the general theory of the evolution equations in Banach spaces (see [1]) we deduce

Theorem. Suppose the assumptions (H.1) and (H.2) hold. If $\begin{pmatrix} i_0 \\ v_0 \end{pmatrix} \in X$ and $\begin{pmatrix} f \\ g \end{pmatrix} \in L^1(0, T; X)$ then the problem (S), (BC), (IC) has a unique integral solution $\begin{pmatrix} i \\ v \end{pmatrix} \in C([0, T]; X)$, which depends Lipschitz continuously on the data of the problem $\begin{pmatrix} i_0 \\ v_0 \end{pmatrix}$ and $\begin{pmatrix} f \\ g \end{pmatrix}$.

Moreover, if $\begin{pmatrix} i_0 \\ v_0 \end{pmatrix} \in D(A)$ and $\begin{pmatrix} f \\ g \end{pmatrix} \in BV(0, T; X)$ then $\begin{pmatrix} i \\ v \end{pmatrix}$ is Lipschitz continuous in $[0, T]$.

3. Proofs

Proof of Lemma 1. We assume without loss of generality that the operators f_1 and f_2 are single-valued. To show that the operator A is accretive we shall prove that

$$\left[\begin{pmatrix} i \\ v \end{pmatrix} - \begin{pmatrix} \bar{i} \\ \bar{v} \end{pmatrix}, A \begin{pmatrix} i \\ v \end{pmatrix} - A \begin{pmatrix} \bar{i} \\ \bar{v} \end{pmatrix} \right]_X \geq 0, \quad \forall \begin{pmatrix} i \\ v \end{pmatrix}, \begin{pmatrix} \bar{i} \\ \bar{v} \end{pmatrix} \in D(A),$$

where

$$\left[\begin{pmatrix} i \\ v \end{pmatrix}, \begin{pmatrix} \bar{i} \\ \bar{v} \end{pmatrix} \right]_X = \lim_{\lambda \searrow 0} \frac{1}{\lambda} \left(\left\| \begin{pmatrix} i \\ v \end{pmatrix} + \lambda \begin{pmatrix} \bar{i} \\ \bar{v} \end{pmatrix} \right\| - \left\| \begin{pmatrix} i \\ v \end{pmatrix} \right\| \right).$$

Let $\begin{pmatrix} i \\ v \end{pmatrix}, \begin{pmatrix} \bar{i} \\ \bar{v} \end{pmatrix} \in D(A)$. Then, we have

$$\begin{aligned} E &= \lim_{\lambda \searrow 0} \frac{1}{\lambda} \left(\left\| \begin{pmatrix} i - \bar{i} \\ v - \bar{v} \end{pmatrix} + \lambda \begin{pmatrix} v' - \bar{v}' \\ i' - \bar{i}' \end{pmatrix} \right\| - \left\| \begin{pmatrix} i - \bar{i} \\ v - \bar{v} \end{pmatrix} \right\| \right) = \\ &= \int_0^1 \lim_{\lambda \searrow 0} \frac{1}{\lambda} \left\{ \max \left[|i - i' + \lambda(v' - \bar{v}')|, |v - \bar{v} + \lambda(i' - \bar{i}')| \right] - \max \left[|i - \bar{i}|, |v - \bar{v}| \right] \right\} dx = \\ &= \int_{A_0} \lim_{\lambda \searrow 0} \frac{1}{\lambda} \left[|i - \bar{i}' + \lambda(v' - \bar{v}')| - |i - \bar{i}| \right] dx + \int_{B_0} \lim_{\lambda \searrow 0} \frac{1}{\lambda} \left[|v - \bar{v} + \lambda(i' - \bar{i}')| - |v - \bar{v}| \right] dx, \end{aligned}$$

where $A_0 = \{x; x \in [0, 1], |i - \bar{i}| \geq |v - \bar{v}|\}$, $B_0 = [0, 1] - A_0$.

Therefore we deduce

$$E = \int_{A_0} \operatorname{sgn}[(i - \bar{i})(v - \bar{v})] |v - \bar{v}| dx + \int_{B_0} \operatorname{sgn}[(i - \bar{i})(v - \bar{v})] |i - \bar{i}| dx.$$

We decompose the sets A_0 and B_0 in a countable reunion of intervals where $[i - \bar{i} > 0 \text{ or } < 0]$ and $[v - \bar{v} > 0 \text{ or } < 0]$. We find

$$\begin{aligned} E &= \operatorname{sgn}[(i - \bar{i})(1)(v - \bar{v})(1)] |u(1) - \bar{u}(1)| - \\ &\quad - \operatorname{sgn}[(i - \bar{i})(0)(v - \bar{v})(0)] |w(0) - \bar{w}(0)|, \end{aligned}$$

with $[u = i \text{ or } v]$ and $[w = i \text{ or } v]$. Using the conditions (BC) we obtain that $E \geq 0$. Hence A is accretive in the space X . q.e.d.

P r o o f of Lemma 2. We suppose, again, that f_1 and f_2 are single-valued. Because A is accretive, to prove that A is m -accretive it suffices to show that for every $\begin{pmatrix} f \\ g \end{pmatrix} \in X$ there exists $\begin{pmatrix} i \\ v \end{pmatrix} \in D(A)$ such that

$$\begin{pmatrix} i \\ v \end{pmatrix} + A \begin{pmatrix} i \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

or, equivalently

$$(1) \quad \begin{cases} i + v' = f, \\ v + i' = g, \end{cases}$$

with $i, v \in W^{1,1}(0,1)$, $i(0) \in -f_1(v(0))$, $i(1) \in f_2(v(1))$.

Using a similar idea as that we used in the proof of Lemma 1.1 [3] we shall look for solutions of the problems (1) of the form $i = i_1 + i_2$, $v = v_1 + v_2$, where $\begin{pmatrix} i_1 \\ v_1 \end{pmatrix} \in (W^{1,1}(0,1))^2$ is a solution of the problem

$$(2) \quad \begin{cases} i_1 + v_1' = f, \\ v_1 + i_1' = g, \\ v_1(0) = v_1(1) = 0 \end{cases}$$

and $\begin{pmatrix} i_2 \\ v_2 \end{pmatrix} \in (W^{1,1}(0,1))^2$ is a solution of the problem

$$(3) \quad \begin{cases} i_2 + v_2' = 0, \\ v_2 + i_2' = 0, \\ \begin{cases} i_2(0) = -f_1(v_2(0)) - i_1(0), \\ i_2(1) = f_2(v_2(1)) - i_1(1). \end{cases} \end{cases}$$

The problem (2) has a unique solution $\begin{pmatrix} i_1 \\ v_1 \end{pmatrix} \in (W^{1,1}(0,1))^2$, which can be founded by the method of "variation of constants".

To solve the problem (3) we deduce first the general solution $\begin{pmatrix} i_2 \\ v_2 \end{pmatrix}$ of the system (3)₁:

$$\begin{cases} i_2(x) = Ae^x + Be^{-x}, \\ v_2(x) = -Ae^x + Be^{-x}, \quad A, B \in \mathbb{R}. \end{cases}$$

We shall fix the constants A and B such that the conditions $(3)_2$ shall be satisfied, that is

$$(4) \quad \begin{cases} A + B = -f_1(-A + B) - i_1(0), \\ Ae + Be^{-1} = f_2(-Ae + Be^{-1}) - i_1(1), \end{cases}$$

where i_1 is the solution of the problem (2).

Let us denote by

$$\begin{cases} -A + B = \alpha, \\ -Ae + Be^{-1} = \beta. \end{cases}$$

Then, we obtain

$$A = \frac{\alpha - \beta e}{e^2 - 1}, \quad B = \frac{\alpha e^2 - \beta e}{e^2 - 1}$$

and the conditions (4) are equivalent to

$$(5) \quad \begin{cases} \frac{\alpha e^2 - 2\beta e + \alpha}{e^2 - 1} + f_1(\alpha) = -i_1(0), \\ \frac{\beta e^2 - 2\alpha e + \beta}{e^2 - 1} + f_2(\beta) = i_1(1). \end{cases}$$

We denote by P the matrix

$$P = \begin{pmatrix} \frac{e^2 + 1}{e^2 - 1} & \frac{-2e}{e^2 - 1} \\ \frac{-2e}{e^2 - 1} & \frac{e^2 + 1}{e^2 - 1} \end{pmatrix}$$

and define the operator $G : D(G) = D(f_1) \times D(f_2) \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$G \begin{pmatrix} i \\ v \end{pmatrix} = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \alpha \in f_1(i), \beta \in f_2(v) \right\}.$$

Under the assumption (H.2) we can easily show that the operator G is maximal monotone in $\mathbb{R}^2$, endowed with standard inner product.

Using the matrix P and the operator G the conditions (5) are equivalent to

$$(6) \quad P \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + G \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -i_1(0) \\ i_1(1) \end{pmatrix}.$$

The operator defined by P is monotone, continuous and coercive. Because G is maximal monotone, we deduce that $P + G$ is a maximal monotone and coercive

operator in $\mathbb{R}^2$. Therefore there exists $\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)$ the unique solution of the Eq. (6) and, equivalently, there exists a unique solution for the problem (3).

The sum of the solutions of the problems (2) and (3) is solution for the problem (1). Thus, the operator A is m -accretive in the space X . q.e.d.

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REFERENCES

1. Barbu V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*. Noordhoff, Leyden, 1976.
2. Luca R., *An Existence Result for a Nonlinear Hyperbolic System*. Diff. Integral Equations, 8, 887-900 (1995).
3. Luca R., *Probleme la limită pentru sisteme cu derivate parțiale de tip hiperbolic și aplicații*. Teză de doctorat, Univ. „Al. I. Cuza”, Iași, 1995.

OBSERVAȚII ASUPRA UNEI PROBLEME HIPERBOLICE

(Rezumat)

Se deduce un rezultat de existență și unicitate pentru soluțiile integrale ale problemei (S), (BC), (IC) în spațiul $C([0, 1]; (L^1(0, 1))^2)$.

BASIC THEOREMS FOR NONSIMPLE THERMOELASTIC SOLIDS

BY

VALERIA BARTILOMO and FRANCESCA PASSARELLA

Abstract. The linear theory of heat-flux dependent thermoelasticity for non-simple bodies is considered. An equation of energy balance, a theorem on the uniqueness of solution and variational and reciprocal results are deduced for centrosymmetric materials.

Key words: nonsimple thermoelastic solid, energy equation, variational theorem, reciprocal relation.

1. Introduction

The general theory of nonsimple elastic materials was studied in several works (see, for example [1]...[6]). The equations of motion and constitutive relations of this theory were firstly given by Toupin [1], [2].

The theory of *nonsimple thermoelastic bodies* has been studied in various paper (see, for example [7]...[10]). In [11] Ciarella studies the general theory of thermoelasticity for non simple materials starting by the results established by Lebon [12] for the case of simple materials. In [11], [12] the theory of thermoelasticity is characterized by the inclusion of the *heat flux vector* in the set of the independent constitutive variables. In the context of linear theory, this inclusion implies that the heat transport equation is of the hyperbolic type and, consequently, the heat signal propagation has a finite speed (*second sound effect*).

The object of this paper is to formulate some basic theorems for the linear generalized (heat-flux dependent) thermoelasticity for centrosymmetric, homogeneous and isotropic non simple bodies. In Sec. 2 we present the basic equations in the context of the results derived in [14]. In Sec. 3 is deduced the equation of energy balance in terms of the Biot's potential [13]. Subsequently, we establish a uniqueness theorem for the initial-mixed boundary value problem. Finally (Sec. 4), we prove two variational theorems, of the Hamilton type and of the Biot type (following [14], [15]) and a reciprocal relation.

2. Basic Equations

We consider a nonsimple body that at time $t = 0$ occupies a region Ω of the physical 3-dimensional space ($\equiv \mathbb{R}^3$) and is bounded by the smooth surface $\partial\Omega$. The motion of the body is referred to a fixed orthonormal frame in $\mathbb{R}^3$ and the time interval in concern is $I = [0, t_0)$, where t_0 can also be infinite.

We shall denote the tensor components of order $p \geq 1$ by Latin subscripts, ranging over $\{1, 2, 3\}$. Summation over repeated subscripts is implied. Superposed dots or subscripts preceded by a comma will mean partial derivative with respect to the time or the corresponding coordinates.

In the context of the linear theory, the behaviour of a nonsimple thermoelastic body is governed by the following local balances of momentum and energy

$$(1) \quad \tau_{ji,j} - \mu_{sji,sj} + \varrho f_i = \varrho \ddot{u}_i,$$

$$q_{i,i} + \varrho r = \varrho \theta_0 \dot{\eta} \quad \text{on } \Omega \times \overset{\circ}{I},$$

where $\mu_{sji} = \mu_{jsi}$ is the *hyperstress* tensor, closely related to the non simple material structure [2]; $\tau_{ji} \equiv T_{ji} + \mu_{sji,s} = \tau_{ji}$ while T_{ji} is the (Cauchy) stress tensor; ϱ is the mass density and f_i is the body force per unit mass; q_i is the heat flux vector within the body; η and r are the entropy and the heat supply fields unit mass, respectively. Moreover, u_i represents the displacement field and $\theta_0 (> 0)$ is the (absolute) temperature in the initial state. We shall denote by θ the actual temperature of the body (measured from θ_0).

According to the classical interpretation of system (1), we shall assume that

- i. $u_i, \theta \in C^{2,2}(\bar{\Omega} \times I)$;
- ii. $\tau_{ji} \in C^{1,0}(\bar{\Omega} \times I)$; $\mu_{sji} \in C^{2,0}(\bar{\Omega} \times I)$; $q_i \in C^{1,1}(\bar{\Omega} \times I)$;
- iii. $\varrho \in C^0(\bar{\Omega})$; $f_i, r \in C^{0,0}(\bar{\Omega} \times I)$.

We restrict our attention to centrosymmetric, homogeneous and isotropic body and assume that the initial bodies is free from stress and hyperstress. In the theory at hand, the independent constitutive variables are

$$(2) \quad E_{ji} \equiv \frac{1}{2} (u_{i,j} + u_{j,i}), \quad K_{sji} \equiv u_{i,sj} = K_{jsi}, \quad \theta \quad \text{and} \quad q_i,$$

and the system of field equations is completed by the following constitutive equations

(see e.g. [11]):

$$\tau_{ji} = \lambda E_{rr} \delta_{ji} + 2\mu E_{ji} - \beta \theta \delta_{ji},$$

$$\begin{aligned} \mu_{sji} = \frac{1}{2} \alpha_1 (K_{rrs} \delta_{ji} + 2K_{irr} \delta_{sj} + K_{rrj} \delta_{si}) + \alpha_2 (K_{srr} \delta_{ji} + K_{jrr} \delta_{si}) + \\ (3) \quad + 2\alpha_3 K_{rri} \delta_{sj} + 2\alpha_4 K_{sji} + \alpha_5 (K_{ijs} + K_{isj}), \end{aligned}$$

$$\varrho \eta = \beta E_{ii} + \alpha \theta,$$

$$\tau \dot{q}_i = k \theta_{,i} - q_i,$$

where δ_{ji} is the Kronecker's delta, λ and μ are the *Lamé constants*, $\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are the *constitutive moduli*. The coefficient τ and k are the *thermal relaxation time* and *thermal conductivity*, respectively; of course, letting $\tau \rightarrow 0$ in (3₄) gives the classical Fourier law of heat conduction.

Inserting the constitutive equations (3_{1,2,3}) and Eqs. (2_{1,2}) in the field Eqs. (1), along with Eq. (3₄), we obtain the following coupled system in the unknowns (u_i, θ, q_i) :

$$\begin{aligned} (\lambda + \mu) u_{j,ji} + \mu u_{i,jj} - \beta \theta_{,i} - 2(\alpha_1 + \alpha_2 + \alpha_5) u_{j,jrri} - \\ - 2(\alpha_3 + \alpha_4) u_{i,rrjj} + \varrho f_i = \varrho \ddot{u}_i, \\ (4) \quad q_{i,i} + \theta_0 \left[\frac{\varrho}{\theta_0} r - \beta \dot{u}_{i,i} - \alpha \dot{\theta} \right] = 0, \\ q_i + \tau \dot{q}_i = k \theta_{,i}. \end{aligned}$$

If we apply the operator $(1 + \tau \frac{\partial}{\partial t})$ to the energy equation and (4₂) and eliminate q_i by Eq. (4₃), we get an evolution equation for θ

$$(5) \quad k \theta_{,ii} + \theta_0 \left(1 + \tau \frac{\partial}{\partial t} \right) \left[\frac{\varrho}{\theta_0} r - \beta \dot{u}_{i,i} - \alpha \dot{\theta} \right] = 0;$$

if we assume that α, τ and k are strictly positive, the differential Eq. (5) admits hyperbolic (wave-type) character and, consequently, the heat signals propagation have the finite speed $(k/\theta_0 \tau \alpha)^{1/2}$ (this is the so-called *second sound effect*).

Consider now three pairs of disjoint and complementary subsets of $\partial\Omega$ denoted by $\{\Sigma_i, \Sigma_{i+1}\}$ with $i = 1, 3, 5$. We define the normal derivate $Df = f_{,i} n_i$, f being any scalar field and n_i is the outward unit normal to $\partial\Omega$.

To the system of field Eqs. (4) we add the following initial-boundary conditions:

$$\begin{aligned} (6) \quad u_i = u_i^0, \quad \dot{u}_i = v_i^0, \quad \eta = \eta^0, \quad q_i = q_i^0, \quad \text{on } \Omega \times \{0\}; \\ u_i = u_i^* \quad \text{on } \Sigma_1 \times I, \quad P_i = P_i^* \quad \text{on } \Sigma_2 \times I, \end{aligned}$$

$$(7) \quad \begin{aligned} Du_i &= g_i^* & \text{on } \Sigma_3 \times I, & & R_i &= R_i^* & \text{on } \Sigma_4 \times I, \\ \theta &= \theta^* & \text{on } \Sigma_5 \times I, & & q &= q^* & \text{on } \Sigma_6 \times I, \end{aligned}$$

where $q = q_j n_j$ is the heat flux associated with the surface $\partial\Omega$; P_i and R_i are two suitably defined vectors such that

$$(8) \quad \int_{\partial\Omega} (t_i \dot{u}_i + \mu_{ji} \dot{u}_{i,j}) d\sigma = \int_{\partial\Omega} (P_i \dot{u}_i + R_i D\dot{u}_i) d\sigma,$$

in which $\mu_{ji} \equiv \mu_{sji} n_s$ and $t_i \equiv (\tau_{ji} - \mu_{sji,s}) n_j$ are the surface hypertraction and traction, respectively. The vectors P_i and R_i are [3]

$$(9) \quad \begin{aligned} P_i &= [\tau_{ji} - \mu_{sji,s} + (\mu_{sji} n_s n_r - m u_{srj} n_s n_j)_{,r}] n_j, \\ R_i &= \mu_{sji} n_s n_j. \end{aligned}$$

The terms on right-hand in (6) and (7) stand for (sufficiently smooth) assigned fields: along with f_i and r , these are the (external) *data* of the mixed problem in concern. An array field (u_i, θ, q_i) meeting all equations (4), (6), (7), or some assignment of the data, will be referred to as a (regular) *solution* of this.

For what follows, it is useful to introduce the *Helmholtz's free energy* ψ defined (see e.g. [11]) by

$$(10) \quad \begin{aligned} \psi &= \frac{1}{2} \lambda E_{rr} E_{ss} + \mu E_{ji} E_{ji} + \alpha_1 K_{ssi} K_{ijj} + \alpha_2 K_{sjj} K_{sii} + \alpha_3 K_{ssi} K_{jji} + \\ &+ \alpha_4 K_{sji} K_{sji} + \alpha_5 K_{sji} K_{ijs} + \frac{1}{2} b q_i q_i - \beta E_{rr} \theta - \frac{1}{2} \alpha \theta^2. \end{aligned}$$

By the Second Principle of Thermodynamics, the following restrictions on the constitutive constants can also be proved [11]:

$$(11) \quad \frac{b}{\tau} k = \frac{1}{\theta_0} \quad \text{and} \quad \frac{b}{\tau} > 0;$$

these restrictions imply that k is, also, strictly positive.

3. Energy Equation and Uniqueness of Solution

Of interest in the sequel will be the following quadratic form

$$(12) \quad \begin{aligned} B(\xi_{ji}, \xi_{sji}, \xi, \xi_i) &= \frac{1}{2} \lambda \xi_{rr} \xi_{ss} + \mu \xi_{ji} \xi_{ji} + \alpha_1 \xi_{ssi} \xi_{ijj} + \alpha_2 \xi_{sjj} \xi_{sii} + \alpha_3 \xi_{ssi} \xi_{jji} + \\ &+ \alpha_4 \xi_{sji} \xi_{sji} + \alpha_5 \xi_{sji} \xi_{ijs} + \frac{1}{2} b \xi_i \xi_i + \frac{1}{2} \alpha \xi^2. \end{aligned}$$

We can note that, for $\xi_{ji} = E_{ji}$, $\xi_{sji} = K_{sji}$, $\xi = \theta$, $\xi_i = q_i$, this form $B \equiv B(E_{ji}, K_{sji}, \theta, q_i)$ attains the meaning of a potential energy ("Biot's potential", see e.g. [13]) to the fields (u_i, θ, q_i) .

Theorem 1. Assume that the relations (11) hold. Then the following equation holds:

$$(13) \quad \frac{d}{dt} \int_{\Omega} (K + B) dV = \int_{\Omega} \left(\varrho f_i \dot{u}_i + \varrho \theta_0 r \theta - \frac{b}{\tau} q_i q_i \right) dV + \\ + \int_{\partial\Omega} \left(P_i \dot{u}_i + R_i D \dot{u}_i + \frac{\theta}{\theta_0} q \right) d\sigma,$$

in which

$$(14) \quad K = \frac{1}{2} \varrho \dot{u}_i \dot{u}_i$$

is the "kinetic energy".

Proof. By Eqs. (3), (14), (12), (4), (8) and divergence theorem we obtain

$$(15) \quad \frac{d}{dt} \int_{\Omega} (K + B) dV = \int_{\Omega} \left(\varrho f_i \dot{u}_i + \varrho \theta_0 r \theta - \frac{1}{\theta_0} q_i \theta_{,i} + b q_i \dot{q}_i \right) dV + \\ + \int_{\partial\Omega} \left(P_i \dot{u}_i + R_i D \dot{u}_i + \frac{\theta}{\theta_0} q \right) d\sigma.$$

Multiplying Eqs. (4₃) by $(b/\tau)q_i$ and using (11), gives

$$\frac{b}{\tau} q_i q_i + b q_i \dot{q}_i = \frac{1}{\theta_0} q_i \theta_{,i},$$

thus Eq. (15) yields Eq. (13).

Eq. (13) is the balance equation of the full thermomechanical energy that involves all independent (constitutive) fields (u_i, θ, q_i) of the theory concerned.

Theorem 1 straightly leads to the following result:

Uniqueness Theorem. Assume that

- i. ϱ, τ, α are strictly positive;
- ii. the relations (11) hold;
- iii. the quadratic form $B(\xi_{ij}, \xi_{ijk}, \xi, \xi_i)$ of (12) is positive semi-definite.

Then, there exists at most one regular solution for the problem (4), (6), (7).

Proof. Thanks to the linearity of the problem, we only need to show that null data imply null solution. Let $\mathcal{U} = (u_i, \theta, q_i)$ a solution corresponding to null data. Of course it holds

$$\theta = 0 \quad \text{on } \Omega \times \{0\},$$

by virtue of Eq. (3₃). Moreover, for null data Eq. (13) reduces to

$$\frac{d}{dt} \int_{\Omega} (K + B) dV = - \int_{\Omega} \frac{b}{\tau} q_i q_i dV$$

that is non-positive by Eq. (11₂) and, accordingly, $K + B$ is a not-increasing function of t . We can also prove, by Eqs. (12), (14) and homogeneous initial conditions, that

$$K = 0, \quad B = 0 \quad \text{on } \Omega \times \{0\},$$

thus

$$K + B \leq 0, \quad \text{on } \Omega \times I.$$

By definitions and hypothesis, K and B are non-negative and, therefore, we obtain

$$(16) \quad K = 0, \quad B = 0 \quad \text{on } \Omega \times I.$$

Since $\rho > 0$ and u_i vanish initially, Eq. (16₁) leads to

$$(17) \quad u_i = 0 \quad \text{on } \Omega \times I.$$

Considering that $\alpha > 0, \tau > 0$, the relation (11₂) holds, from Eqs. (12), (16₂), (17) we have

$$(18) \quad \theta = 0, \quad q_i = 0 \quad \text{on } \Omega \times I.$$

Eqs. (17), (18) give us the desired result.

4. Variational Theorems and Reciprocal Relation

In this section, following [15], we establish two variational theorems, one of Hamilton type and one of Biot type, and a reciprocal relation.

Variational Principle I. Let $t_1 \leq t_0$ and $t_2 \leq t_0$ be any instants. Then, for arbitrary variations in u_i and θ such that

$$(19) \quad \delta u_i = 0, \quad \delta \theta = 0, \quad \text{in } \Omega \times \{t_1, t_2\};$$

$$(20) \quad \begin{aligned} \delta u_i &= 0 \quad \text{on } \Sigma_1 \times [t_1, t_2], & \delta D u_i &= 0 \quad \text{on } \Sigma_3 \times [t_1, t_2], \\ \delta \theta &= 0 \quad \text{on } \Sigma_5 \times [t_1, t_2], \end{aligned}$$

with the functions $f_i, \eta, r, P_i^*, R_i^*, q^*, t_1$ and t_2 being kept unchanged, the following variational equations

$$(21) \quad \begin{aligned} & \delta \int_{t_1}^{t_2} dt \left[\int_{\Omega} \left(\frac{1}{2} \rho \dot{u}_i \dot{u}_i - \psi - \rho \eta T + \frac{1}{2} b q_i q_i + \rho f_i u_i \right) dV + \right. \\ & \left. + \int_{\Sigma_2} P_i^* u_i d\sigma + \int_{\Sigma_4} R_i^* D u_i d\sigma \right] = 0, \end{aligned}$$

$$(22) \quad \delta \int_{t_1}^{t_2} dt \left[\int_{\Omega} \frac{1}{2} k T_{,i} T_{,i} dV + \right. \\ \left. - (1 + \tau p) \left\{ \int_{\Omega} \varrho (\eta \dot{T} + r) T dV + \int_{\Sigma_6} q^* T u_i d\sigma \right\} \right] = 0,$$

are satisfied if and only if are verified, through Eqs. (7), (9), the field equations (4₁) and (5).

In the above equations we put $T = \theta + \theta_0$ (absolute temperature) and $p \equiv \partial/\partial t$.

P r o o f. Under the conditions (19₁), we get

$$(23) \quad \delta \int_{t_1}^{t_2} dt \int_{\Omega} \frac{1}{2} \varrho \dot{u}_i \dot{u}_i dV = - \int_{t_1}^{t_2} dt \int_{\Omega} \varrho \ddot{u}_i \delta u_i dV.$$

Moreover, using Eqs. (10), (2), (3) and the divergence theorem, we can obtain

$$(24) \quad \delta \int_{t_1}^{t_2} dt \int_{\Omega} \left(\psi + \varrho \eta T - \frac{1}{2} b q_i q_i \right) dV = \\ = \int_{t_1}^{t_2} dt \int_{\Omega} (\tau_{ji} \delta E_{ji} + \mu_{rji} \delta K_{rji}) dV = \\ = \int_{t_1}^{t_2} dt \left[\int_{\Omega} -(\tau_{ji,i} - \mu_{sjj,sj}) \delta u_i dV + \right. \\ \left. + \int_{\partial\Omega} \{ (\tau_{ji} - \mu_{sjj,s}) \delta u_i n_j + \mu_{sjj} \delta u_{i,j} n_s \} d\sigma \right].$$

With the aid of Eqs. (7), (9) and of hypothesis (20_{1,2}) we can write Eq. (24) as

$$(25) \quad \delta \int_{t_1}^{t_2} dt \int_{\Omega} \left(\psi + \varrho \eta T - \frac{1}{2} b q_i q_i \right) dV = \\ = \int_{t_1}^{t_2} dt \left[\int_{\Omega} -(\tau_{ji,i} - \mu_{sjj,sj}) \delta u_i dV + \int_{\Sigma_2} P_i^* \delta u_i d\sigma + \int_{\Sigma_4} R_i^* D \delta u_i d\sigma \right].$$

Taking into account Eqs. (23), (25), (2) (3_{1,2}), the variational equation (21) becomes

$$\int_{t_1}^{t_2} dt \int_{\Omega} [-\rho \ddot{u}_i + (\lambda + \mu) u_{j,ji} + \mu u_{i,jj} - \beta \theta_{,i} - 2(\alpha_1 + \alpha_2 + \alpha_5) u_{j,jrri} - \\ - 2(\alpha_3 + \alpha_4) u_{i,rrjj} + \rho f_i] \delta u_i dV = 0.$$

This equation holds for any δu_i if and only if Eq. (4₁) are satisfied.

From Eqs. (3₄) and (20₃) it follows that

$$\delta \int_{t_1}^{t_2} dt \int_{\Omega} \frac{1}{2} k T_{,i} T_{,i} dV = \int_{t_1}^{t_2} dt \left[(1 + \tau p) \int_{\Sigma_6} q^* \delta \theta d\sigma - \int_{\Omega} \theta_{,ii} \delta \theta dV \right],$$

where p has been treated as a constant [13].

Thus, Eq. (22) can be written as

$$\int_{t_1}^{t_2} dt \int_{\Omega} \{k \theta_{,ii} + \rho(1 + \tau p)(r - T \dot{\eta})\} \delta \theta dV = - \int_{\Omega} (\eta T \delta T)|_{t_1}^{t_2} dV.$$

After the linearization, from Eqs. (19), (3₃) we conclude that

$$\int_{t_1}^{t_2} dt \int_{\Omega} \{k \theta_{,ii} + (1 + \tau p) (\rho r - \theta_0 \beta \dot{u}_{i,i} - \theta_0 \alpha \dot{\theta})\} \delta \theta dV = 0.$$

This equation holds if and only if Eq. (5) is satisfied.

We can establish another variational principle and therefore, following [13], we introduce the *entropy flow* vector s_i by

$$\theta_0 \dot{s}_i = q_i,$$

and we consider prescribed s_i on Σ_6 since q_i is assigned on Σ_6 (see. Eqs. (7)). Moreover, we can rewrite Eq. (4₂) as

$$(26) \quad \dot{s}_{i,i} = \left(\beta \dot{u}_{i,i} + \alpha \dot{\theta} - \frac{\rho}{\theta_0} r \right).$$

We can prove that

Variational Principle II. Let f_i and r be prescribed in Ω , and P_i, R_i , and θ on $\Sigma_2, \Sigma_4, \Sigma_5$, respectively (as in Eqs. (7)). Let the relation (11) hold and define

$$V = \int_{\Omega} (\psi + \varrho \eta \theta - \frac{1}{2} b q_i q_i) dV,$$

$$G = - \int_{\Sigma_2} P_i^* u_i d\sigma - \int_{\Sigma_4} R_i^* D u_i d\sigma - \int_{\Sigma_5} \theta^* s_i n_i d\sigma,$$

$$F = \int_{\Omega} \varrho \left(\frac{1}{2} p^2 u_i - f_i \right) u_i dV,$$

$$D = \frac{1}{2} \int_{\Omega} \theta_0 (p + \tau p^2) \frac{1}{k} s_i s_i dV.$$

Then, for arbitrary variations $\delta u_i (= 0$ on Σ_1), $\delta D u_i (= 0$ on Σ_3), $\delta \theta (= 0$ on Σ_5), $\delta s_i (= 0$ on Σ_6) the following variational equation

$$\delta H(u_i, \theta) = \delta V + \delta G + \delta F + \delta D = 0$$

yields the field Eqs. (4₁), (5).

In above equations, the symbol p is used as before.

P r o o f. By Eqs. (10), (3), (9), we get

$$(27) \quad \delta V = \int_{\Omega} \{ \alpha \theta \delta \theta + (\tau_{ji} - \beta \theta \delta_{ij}) \delta u_{i,j} + \mu_{sji} \delta u_{i,sj} \} dV.$$

By hypothesis, Eqs. (7), (9) and divergence theorem we have

$$\begin{aligned} \delta G &= - \int_{\partial \Omega} \{ P_i \delta u_i + R_i D \delta u_i + \theta n_i \delta s_i \} d\sigma = \\ &= - \int_{\partial \Omega} \{ (\tau_{ji} - \mu_{sji,s}) \delta u_i n_j + \mu_{sji} \delta u_{i,j} n_s + \theta n_i \delta s_i \} d\sigma = \\ (28) \quad &= - \int_{\Omega} \{ \alpha \theta \delta \theta + (\tau_{ji} + \beta \theta \delta_{ij}) \delta u_{i,j} + \mu_{sji} \delta u_{i,sj} \} dV - \\ &\quad - \int_{\Omega} \{ (\tau_{ji,j} - \mu_{sji,sj}) \delta u_i + \theta_{,i} \delta s_i \} dV, \end{aligned}$$

where Eq. (26) has been also used to calculate $\delta s_{i,i}$.

Moreover, we obtain

$$(29) \quad \delta F = \int_{\Omega} \varrho (p^2 u_i - f_i) \delta u_i dV,$$

$$(30) \quad \delta D = \int_{\Omega} \theta_0 (p + \tau p^2) \frac{1}{k} s_i \delta s_i dV.$$

By Eqs. (27)...(30), (3_{1,2}) we can write

$$\begin{aligned} \delta H(u_i, \theta) = \int_{\Omega} \{ [\varrho \ddot{u}_i - \varrho f_i - (\lambda + \mu) u_{j,ji} - \mu u_{i,jj} + \beta \theta_{,i} + 2(\alpha_1 + \alpha_2 + \alpha_5) u_{j,jrri} + \\ + 2(\alpha_3 + \alpha_4) u_{i,rrjj}] \delta u_i + [\theta_0(1 + \tau p) \frac{1}{k} \dot{s}_i - \theta_{,i}] \delta s_i \} dV. \end{aligned}$$

Vanishing of this expression for arbitrary $\delta u_i, \delta s_i$ implies the field Eq. (4₁); further, from the last term in the integral above, we get

$$\theta_0(1 + \tau p) \frac{1}{k} \dot{s}_i = \theta_{,i}.$$

If we multiply the above equation by k and derive with respect to the i -th coordinate, we can easily deduce the field Eq. (5) from Eq. (26). This completes the proof.

In what follows we shall consider the mixed problem with homogeneous initial conditions and denote by

$$\mathcal{T}^{(\alpha)} = \{ f_i^{(\alpha)}, r^{(\alpha)}, u_i^{*(\alpha)}, P_i^{*(\alpha)}, Du_i^{*(\alpha)}, R_i^{*(\alpha)}, \theta^{*(\alpha)}, q^{*(\alpha)} \},$$

$$\mathcal{U}^{(\alpha)} = \{ u_i^{(\alpha)}, \theta^{(\alpha)} \}, \quad \alpha = 1, 2,$$

two different sets of data and the corresponding solutions for the problem (4₁), (5)...(7) respectively. Moreover, we define $\tau_{ij}^{(\alpha)}, \mu_{ijk}^{(\alpha)}, \eta^{(\alpha)}, q_i^{(\alpha)}$ by means of Eqs. (3) for each $\alpha = 1, 2$.

We can prove that

Reciprocal Theorem. Let $\mathcal{U}^{(\alpha)}$ be solutions corresponding to different sets of data $\mathcal{T}^{(\alpha)}$ ($\alpha = 1, 2$). Then, the following relation holds

$$(31) \quad \mathcal{T}_{12} = \mathcal{T}_{21},$$

where

$$\begin{aligned} \mathcal{T}_{\alpha\beta} = \int_{\Omega} \varrho \left[f_i^{(\alpha)} * \dot{u}_i^{(\beta)} - \frac{1}{\theta_0} r^{(\alpha)} * \theta^{(\beta)} \right] dV - \\ - \int_{\Sigma_1} u_i^{*(\alpha)} * \dot{P}_i^{(\beta)} d\sigma + \int_{\Sigma_2} P_i^{*(\alpha)} * \dot{u}_i^{(\beta)} d\sigma - \\ - \int_{\Sigma_3} Du_i^{*(\alpha)} * \dot{R}_i^{(\beta)} d\sigma + \int_{\Sigma_4} R_i^{*(\alpha)} * \dot{Du}_i^{(\beta)} d\sigma - \\ + \frac{1}{\theta_0} \int_{\Sigma_5} \theta^{*(\alpha)} * q^{(\beta)} n_j d\sigma - \frac{1}{\theta_0} \int_{\Sigma_6} q^{*(\alpha)} * \theta^{(\beta)} d\sigma \end{aligned}$$

for $\alpha, \beta = 1, 2$.

In the above equation we have denoted by $a_1 * a_2$ the time convolution of scalar fields on $\Omega \times I$, i.e.

$$a_1 * a_2(\mathbf{x}, t) = \int_0^t a_1(\mathbf{x}, t - \tau) a_2(\mathbf{x}, \tau) d\tau = a_2 * a_1(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times I.$$

P r o o f. We put

$$\begin{aligned} \mathcal{F}_{\alpha, \beta} &= \left((\tau_{ji}^{(\alpha)} - \mu_{sji, s}^{(\alpha)}) * \dot{u}_i^{(\beta)} + \mu_{sji}^{(\alpha)} * \dot{u}_{i, s}^{(\beta)} \right)_{, j} + \varrho f_i^{(\alpha)} * \dot{u}_i^{(\beta)}, \\ \mathcal{H}_{\alpha, \beta} &= \left(q_i^{(\alpha)} * \theta^{(\beta)} \right)_{, i} + \varrho r^{(\alpha)} * \theta^{(\beta)}. \end{aligned}$$

Taking into account that for (any) pair of functions a_1 and a_2 under null initial conditions we have

$$(32) \quad a_1 * \dot{a}_2 = \dot{a}_1 * a_2,$$

the constitutive relation and the equations of motion and of energy imply that

$$\begin{aligned} \mathcal{F}_{\alpha\beta} - \mathcal{F}_{\beta\alpha} &= \beta \left(\dot{u}_{i, i}^{(\alpha)} * \theta^{(\beta)} - \theta^{(\alpha)} * \dot{u}_{i, i}^{(\beta)} \right), \\ \mathcal{H}_{\alpha\beta} - \mathcal{H}_{\beta\alpha} &= \theta_0 \beta \left(\dot{u}_{i, i}^{(\alpha)} * \theta^{(\beta)} - \theta^{(\alpha)} * \dot{u}_{i, i}^{(\beta)} \right). \end{aligned}$$

Thus, we conclude that

$$(33) \quad (\mathcal{F}_{\alpha\beta} - \mathcal{F}_{\beta\alpha}) - \frac{1}{\theta_0} (\mathcal{H}_{\alpha\beta} - \mathcal{H}_{\beta\alpha}) = 0.$$

If we integrate this equation over Ω and use the divergence theorem, we get

$$\begin{aligned} (34) \quad & \int_{\Omega} \left\{ \varrho \left[f_i^{(\alpha)} * \dot{u}_i^{(\beta)} - f_i^{(\beta)} * \dot{u}_i^{(\alpha)} \right] - \frac{\varrho}{\theta_0} \left[r^{(\alpha)} * \theta^{(\beta)} - r^{(\beta)} * \theta^{(\alpha)} \right] \right\} dV + \\ & + \int_{\partial\Omega} \left\{ \left(\tau_{ji}^{(\alpha)} - \mu_{rji, r}^{(\alpha)} \right) * \dot{u}_i^{(\beta)} + \mu_{rji}^{(\alpha)} * \dot{u}_{i, r}^{(\beta)} \right\} n_j d\sigma - \\ & - \int_{\partial\Omega} \left\{ \left(\tau_{ji}^{(\beta)} - \mu_{rji, r}^{(\beta)} \right) * \dot{u}_i^{(\alpha)} + \mu_{rji}^{(\beta)} * \dot{u}_{i, r}^{(\alpha)} \right\} n_j d\sigma + \\ & + \frac{1}{\theta_0} \int_{\partial\Omega} \left\{ \theta^{(\alpha)} * q^{(\beta)} - \theta^{(\beta)} * q^{(\alpha)} \right\} d\sigma = 0, \end{aligned}$$

so that from Eqs. (9), (32), (34) and from the boundary conditions we obtain Eq. (31).

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REFERENCES

1. Toupin R. A., *Elastic Materials with Couple-Stresses*. Arch. Rational Mech. Anal., **11**, 385-414 (1962).
2. Toupin R. A., *Theories of Elasticity with Couple-Stresses*. Arch. Rational Mech. Anal., **17**, 85-112 (1964).
3. Mindlin R. D., *Microstructure in Linear Elasticity*. Arch. Rational Mech. Anal., **16**, 51-78 (1964).
4. Mindlin R. D., Eshel N. N., *On First Strain Gradient Theories in Linear Elasticity*. Int. J. Solids Struct., **44**, 109-124 (1968).
5. Lardner R. W., *The Linear Theory of Second-Grade Elastic Materials*. Q. Appl. Math., **27**, 323-334 (1969).
6. Ciarletta M., Ieșan D., *Non-classical Elastic Solids*. Pitman Research Notes in Mathematics Series (Ed. by Longman Scientific & Technical), **293**, 1993.
7. Ahmadi G., Firoozbakhsh K., *First Strain-Gradient Theory of Thermoelasticity*. Int. J. Solids Struct., **11**, 339-345 (1975).
8. Ciarletta M., Ieșan D., *On the Nonlinear Theory of Nonsimple Thermoelastic Bodies*. J. Therm. Stresses, **12**, 545-558 (1989).
9. Ieșan D., *Thermoelasticity of Nonsimple Materials*. J. Therm. Stresses, **6**, 167-188 (1983).
10. Ciarletta M., Scalia A., *Thermoelasticity of Nonsimple Materials with Voids*. Rev. Roum. Sci. Tech. Tech. Mech. Appl., **35**, 115-125 (1990).
11. Ciarletta M., *Thermoelasticity of Nonsimple Materials with Thermal Relaxation*. J. Thermal Stresses. (1996) (in print).
12. Lebon G., *A Generalized Theory of Thermoelasticity*. J. Tech. Phys., **23**, 37-46 (1982).
13. Biot M. A., *Thermoelasticity and Irreversible Thermodynamics*. J. Appl. Phys., **7**, 240-255 (1956).
14. Chandrasekharaiah D. S., *Heat-flux Dependent Micropolar Thermoelasticity*. Int. J. Engng. Sci., **24**, 1389-1395 (1986).
15. Chandrasekharaiah D. S., *Variational and Reciprocal Principles in Micropolar Thermoelasticity*. Int. J. Engng. Sci., **25**, 55-63 (1987).

TEOREME DE BAZĂ PENTRU SOLIDE TERMOELASTICE NESIMPLE

(Rezumat)

Se stabilesc un număr de teoreme de bază pentru termoelasticitatea generalizată (dependentă de fluxul termic) liniară, pentru corpuri nesimplete, izotrope, omogene și centrosimetrice. Se prezintă ecuațiile fundamentale și se deduc ecuația de bilanț energetic, o teoremă de unicitate, două teoreme variaționale și o relație de reciprocitate.

ON THE STABILITY OF VISCOELASTIC EQUILIBRIUM

BY

GH. GR. CIOBANU

Abstract. Some of author's results ([3]...[6]) regarding the stability of viscoelastic equilibrium are revised in order to define them more accurately and to present them in a unitary and concise form.

Key words: linear viscoelasticity, equilibrium, stability.

1. The Statement of the Initial and Boundary-Value Problem of Linear Viscoelasticity

We consider a linear, nonhomogeneous, anisotropic and non-convolutive viscoelastic material occupying, in the time interval $I = [0, T]$, $0 < T < \infty$, a bounded and closed regular region [9] B of the three-dimensional Euclidian space referred to a fixed rectangular Cartesian axes. The behaviour of such a viscoelastic material is governed by the constitutive equations

$$\begin{aligned} \sigma_{ij}(\mathbf{x}, t) &= \sigma_{ij}^{(e)}(\mathbf{x}, t) + \sigma_{ij}^{(v)}(\mathbf{x}, t), \\ \sigma_{ij}^{(e)}(\mathbf{x}, t) &= C_{ijkl}(\mathbf{x}, t) u_{k,l}(\mathbf{x}, t), \\ \sigma_{ij}^{(v)}(\mathbf{x}, t) &= \int_0^t G_{ijkl}(\mathbf{x}, t) u_{k,l}(\mathbf{x}, t, \tau) \dot{u}_{k,l}(\mathbf{x}, \tau) d\tau, \end{aligned} \quad (1.1)$$

where $\sigma_{ij}(\mathbf{x}, t)$ are the Cartesian components of the stress tensor σ , $u_i(\mathbf{x}, t)$ are the Cartesian components of the displacements vector $\mathbf{u}$, $C_{ijkl}(\mathbf{x}, t)$ are the *elasticities*, and $G_{ijkl}(\mathbf{x}, t, \tau)$ are the *relaxation moduli* of the material in the point $\mathbf{x} = (x_1, x_2, x_3)$ and at time $t \in I$.

The tensors $\sigma_{ij}^{(e)}$ and $\sigma_{ij}^{(v)}$ are referred to as the *elastic part*, respectively, the *viscous part* of the stress tensor.

All indices appearing in (1.1) range over values 1, 2, 3, the usual convention of summation over repeated indices is adopted, and a comma followed by the index i indicates the differentiation with respect to x_i .

The elasticities and the relaxation moduli are supposed to satisfy the symmetries

$$(1.2) \quad C_{ijkl} = C_{jikl} = C_{klij}; \quad G_{ijkl} = G_{jikl} = G_{klij}.$$

We shall designate by ∂B the boundary of B (∂B is a regular surface [9]) and by $\mathbf{n} = (n_1, n_2, n_3)$ the field of the outward unit normal to ∂B .

Let $\mathbf{F}(\mathbf{x}, t) = (F_1(\mathbf{x}, t), F_2(\mathbf{x}, t), F_3(\mathbf{x}, t))$, $(\mathbf{x}, t) \in B \times I$, be the body force acting on B and $\rho(\mathbf{x}) > 0$, $\mathbf{x} \in B$, the mass density which is supposed to be independent of t .

The initial and boundary-value problem of the linear dynamic viscoelasticity consists in finding a function $\mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$, $(\mathbf{x}, t) \in B \times I$, satisfying, in a sense which has to be specified, the integro-differential system

$$(1.3) \quad \sigma_{ij,j}(\mathbf{x}, t) + \rho(\mathbf{x})F_i(\mathbf{x}, t) = \ddot{u}_i(\mathbf{x}, t), \quad (\mathbf{x}, t) \in B \times I,$$

the boundary conditions

$$(1.4) \quad \begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \mathbf{h}(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \partial B_1 \times I; \\ \sigma_{ij}(\mathbf{x}, t)n_i(\mathbf{x}) &= s_i(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \partial B_2 \times I, \end{aligned}$$

and the initial conditions

$$(1.5) \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0, \quad \dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{v}^0(\mathbf{x}), \quad \mathbf{x} \in B.$$

Here $\dot{\mathbf{u}} = (\dot{u}_1, \dot{u}_2, \dot{u}_3)$ and $\ddot{\mathbf{u}} = (\ddot{u}_1, \ddot{u}_2, \ddot{u}_3)$ denote the first and second derivatives with respect to t holding $\mathbf{x} \in B$ fixed, $\mathbf{u}^0$ and $\mathbf{v}^0$ are à priori given functions on B , $\partial B_1, \partial B_2 \subset \partial B$ are nonvoid open subsets relative to ∂B such that $\partial B_1 \cap \partial B_2 = \emptyset$, $\partial B_1 \cup \partial B_2 = \partial B$ (the situation $\partial B_2 = \emptyset$ is not excluded), and the functions $\mathbf{h} = (h_1, h_2, h_3)$ and $\mathbf{s} = (s_1, s_2, s_3)$ are à priori given on $\partial B_1 \times I$ and $\partial B_2 \times I$, respectively.

2. The Generalized Solution in the Case of Homogeneous Boundary Condition

Because the problem (1.1), (1.5) is linear, the stability of every solution of this problem is determined by the stability of its null equilibrium solution.

If $\mathbf{U}$ and $\mathbf{U} + \mathbf{u}$ are two solutions of the problem (1.1), (1.3)–(1.5), corresponding to the same body force and to the same boundary conditions, and are satisfying the initial data $\mathbf{U}_0, \mathbf{V}_0$ and $\mathbf{U}_0 + \mathbf{u}^0, \mathbf{V}_0 + \mathbf{v}^0$, respectively, then the *perturbation* $\mathbf{u}(\mathbf{x}, t)$ will satisfy the following mixed initial and boundary-value problem:

$$(2.1) \quad \sigma_{ij,j}(\mathbf{x}, t) = \rho(\mathbf{x})\ddot{u}_i(\mathbf{x}, t), \quad (\mathbf{x}, t) \in B \times I,$$

$$(2.2) \quad \begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \mathbf{0}, \quad (\mathbf{x}, t) \in B_1 \times I, \\ \sigma_{ij}(\mathbf{x}, t) n_i(\mathbf{x}) &= 0, \quad (\mathbf{x}, t) \in \partial B_2 \times I, \end{aligned}$$

$$(2.3) \quad \mathbf{u}(\mathbf{x}, t) = \mathbf{u}^0(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{v}^0(\mathbf{x}), \quad \mathbf{x} \in B.$$

By $C_0^\infty(B)$ it is denoted the set of vector functions

$$\mathbf{x} \rightarrow \mathbf{f}(\mathbf{x}) = (f_1(x), f_2(x), f_3(x)) \in \mathbb{R}^3, \quad \mathbf{x} \in B,$$

having the components f_i in $C_0^\infty(B)$. Let $\mathbf{H}_0$ and $\mathbf{H}_+$ be the Hilbert spaces obtained by the closure of $C_0^\infty(B)$ in the norms $\|\cdot\|_0$ and $\|\cdot\|_+$ induced by the scalar products

$$\langle \mathbf{u}, \mathbf{v} \rangle_0 = \int_B u_i v_i dx, \quad \langle \mathbf{u}, \mathbf{v} \rangle_+ = \int_B u_{i,j} v_{i,j} dx,$$

respectively, and let $\mathbf{H}_-$ be the closure $C_0^\infty(B)$ with respect to the norm

$$\|\mathbf{u}\|_- = \sup_{\mathbf{v} \in \mathbf{H}_+} \frac{|\langle \mathbf{u}, \mathbf{v} \rangle_0|}{\|\mathbf{v}\|_+}.$$

Definition [7]. By a generalized solution of the problem (1.1), (2.1)–(2.3) we mean a function $\mathbf{u} \in L^2(I, \mathbf{H}_+)$ with $\dot{\mathbf{u}} \in L^2(I, \mathbf{H}_0)$ and $\ddot{\mathbf{u}} \in L^2(I, \mathbf{H}_-)$ which satisfies (2.1) almost everywhere and the initial conditions (2.3)₁ and (2.3)₂ in $[\mathbf{H}_+, \mathbf{H}_0]_{1/2}$ and in $[\mathbf{H}_+, \mathbf{H}_0]_{1/2}$ respectively (see Ch. 7 of [1] and Ch. 1 of [2]).

The following theorem is a consequence of a more general results obtained by D a f e r m o s [7], [8].

Theorem 2.1. *There exists a unique generalized solution of the problem (1.1), (2.1)–(2.3), under the following hypotheses:*

(H₁) *The elasticities C_{ijkl} and their time derivatives $\dot{C}_{ijkl}$ are Lebesgue measurable and essentially bounded functions on B for every fixed $t \in I$.*

(H₂) *The relaxation moduli G_{ijkl} and their derivatives $\frac{G_{ijkl}}{\partial \tau}$ are Lebesgue measurable and essentially bounded functions on B for every $(t, \tau) \in I \times I$.*

(H₃) *There exists the strictly positive constant $a > 0$ such that for every $\mathbf{u} \in L^2(I, \mathbf{H}_+)$ we have*

$$(2.4) \quad \int_B C_{ijkl}(\mathbf{x}, t) u_{i,j}(\mathbf{x}, t) u_{k,l}(\mathbf{x}, t) dx \geq a \|\mathbf{u}(\cdot, t)\|_-^2,$$

uniformly with respect to $t \in I$.

(H₄) *The density ρ is essentially bounded on B and there exists $\rho_0 > 0$ such that*

$$(2.5) \quad \text{ess inf}_B \rho(x) \geq \rho_0 > 0.$$

3. A Preliminary Energy Equality

Multiplying (2.1) by $\dot{u}_i$, summing with respect to i from 1 to 3, integrating over B , applying the divergence theorem, and taking into account the boundary conditions (2.2) we get

$$(3.1) \quad \frac{d\tilde{E}}{dt}(t) = \frac{1}{2} \int_B \dot{C}_{ijkl}(\mathbf{x}, t) u_{i,j}(\mathbf{x}, t) u_{k,l}(\mathbf{x}, t) d\mathbf{x} - \int_B \sigma_{ij}^{(v)}(\mathbf{x}, t) \dot{u}_{i,j}(\mathbf{x}, t) d\mathbf{x},$$

where

$$(3.2) \quad \begin{aligned} \tilde{E}(t) &= \frac{1}{2} \int_B [\rho(\mathbf{x}) \dot{u}^2(\mathbf{x}, t) + C_{ijkl}(\mathbf{x}, t) u_{k,l}(\mathbf{x}, t) u_{i,j}(\mathbf{x}, t)] d\mathbf{x} = \\ &= \frac{1}{2} \int_B [\rho(\mathbf{x}) \dot{u}^2(\mathbf{x}, t) + \sigma_{ij}^{(e)}(\mathbf{x}, t) u_{i,j}(\mathbf{x}, t)] d\mathbf{x}, \end{aligned}$$

is the sum of the kinetic energy and of the *elastic energy of deformation*.

By integrating (3.1) on $[0, t] \subset I$ we obtain

$$(3.3) \quad \begin{aligned} \tilde{E}(t) - \tilde{E}(0) &= \int_0^t d\eta \int_B \dot{C}_{ijkl}(\mathbf{x}, \eta) u_{k,l}(\mathbf{x}, \eta) u_{i,j}(\mathbf{x}, \eta) d\mathbf{x} - \\ &- \int_0^t d\eta \int_B \sigma_{ij}^{(v)}(\mathbf{x}, \eta) \dot{u}_{i,j}(\mathbf{x}, \eta) d\mathbf{x}, \quad t \in I, \end{aligned}$$

where

$$(3.4) \quad \tilde{E}(0) = \frac{1}{2} \int_B [\rho(\mathbf{x}) v^{02}(\mathbf{x}) + C_{ijkl}(\mathbf{x}, 0) u_{k,l}^0(\mathbf{x}, 0) u_{i,j}^0(\mathbf{x}, 0)] d\mathbf{x}.$$

Besides the hypotheses (H_1) – (H_4) assuring the existence and uniqueness of the generalized solution of the problem (1.1), (2.1)–(2.3), in the sequel we make the further two basic assumptions in order to obtain some results regarding the stability of the viscoelastic equilibrium in the next section. We formulate now these assumptions.

(A₁) There is a constant $b > 0$ such that for any $\mathbf{u} \in L^2(I; \mathbf{H}_+)$ the inequality

$$(3.5) \quad - \int_B \dot{C}_{ijkl}(\mathbf{x}, t) u_{k,l}(\mathbf{x}, t) u_{i,j}(\mathbf{x}, t) d\mathbf{x} \geq b \|\mathbf{u}(\cdot, t)\|_+^2$$

holds, uniformly with respect to $t \in I$.

(A₂) For any $\mathbf{u} \in L^2(I, \mathbf{H}_+)$ with $\dot{\mathbf{u}} \in L^2(I, \mathbf{H}_-)$ we have

$$(3.6) \quad \int_0^t d\eta \int_B \sigma_{ij}^{(v)}(\mathbf{x}, \eta) \dot{u}_{i,j}(\mathbf{x}, \eta) d\mathbf{x} \leq 0, \quad t \in I.$$

The assumption (A_1) is essential in obtaining some results concerning the stability of equilibrium of an elastic body with time-dependent elasticities (see Sec. 36 of [11]), while inequalities similar to (3.6) have been used by D a f e r m o s [7], [8] in the study of Liapunov asymptotic stability of viscoelastic solutions. The assumption (A_2) shows that the power of the viscous part of the stress tensor is non-positive on any interval $[0, t] \subset I$.

4. Stability of the Viscoelastic Equilibrium [3]–[4]

Using the Theorem 2.1 we may define the *dynamical system* [11] of the linear viscoelasticity as consisting of functions $\mathbf{u} \in L^2(I; \mathbf{H}_+)$ with $\dot{\mathbf{u}} \in L^2(I; \mathbf{H}_0)$ and $\ddot{\mathbf{u}} \in L^2(I; \mathbf{H}_-)$ which satisfy almost everywhere the integro-differential system (1.1), (2.1)

We mention that all the following results are obtained providing the assumptions (A_1) and (A_2) hold.

By taking into account (A_1) and (A_2) , from (3.3) it results

$$(4.1) \quad \tilde{E}(t) \leq \tilde{E}(0), \quad t \in I,$$

and therefore we obtain the

Theorem 4.1. *The viscoelastic equilibrium is uniformly Liapunov stable with respect to the measure [4]*

$$(4.2) \quad \mathcal{E}(\mathbf{u}) = \tilde{E}(t) \quad \text{and} \quad \mathcal{E}_0(\mathbf{u}) = \tilde{E}(0),$$

that is

$$(4.3) \quad \forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon) > 0 : \quad \mathcal{E}_0(\mathbf{u}) < \delta \Rightarrow \mathcal{E}(\mathbf{u}) < \varepsilon, \quad t \in I.$$

By using the hypotheses (H_3) and (H_4) we get the inequality

$$(4.4) \quad \|\dot{\mathbf{u}}(\cdot, t)\|_0^2 + \|\mathbf{u}(\cdot, t)\|_+^2 \leq \frac{2}{m} \tilde{E}(0), \quad m = \min \{\rho_0, a\},$$

and therefore we have the

Theorem 4.2. *The solution of viscoelastic equilibrium problem is uniformly Liapunov stable with respect to the measures*

$$(4.5) \quad \mu(\mathbf{u}) = \|\dot{\mathbf{u}}(\cdot, t)\|_0^2 + \|\mathbf{u}(\cdot, t)\|_+^2 \quad \text{and} \quad \mathcal{E}_0(\mathbf{u}) = \tilde{E}(0).$$

By applying the Schwarz inequality to the integral

$$\int_B C_{ijkl}(\mathbf{x}, 0) u_{i,j}^0(\mathbf{x}) u_{k,l}^0(\mathbf{x}) d\mathbf{x}$$

we obtain

$$(4.6) \quad 2\tilde{E}(0) \leq k \left(\|\dot{\mathbf{u}}(\cdot, t)\|_0^2 + \|\mathbf{u}(\cdot, t)\|_+^2 \right),$$

where $k = \max \left\{ \operatorname{ess\,sup}_{x \in B} \rho(x), \operatorname{ess\,sup}_{x \in B} \sqrt{\operatorname{tr} C^2(\mathbf{x}, 0)} \right\}$ and $C(\mathbf{x}, 0)$ is the symmetric matrix of type 6×6 of elements $C_{ijkl}(\mathbf{x}, 0)$, the indices ij and kl take the values 11, 22, 33, 23, 31, 12.

Therefore, from (4.4) and (4.6) it follows the inequality

$$(4.7) \quad \|\dot{\mathbf{u}}(\cdot, t)\|_0^2 + \|\mathbf{u}(\cdot, t)\|_0^2 \leq \frac{k}{m} \left(\|\dot{\mathbf{u}}(\cdot, t)\|_0^2 + \|\mathbf{u}(\cdot, t)\|_+^2 \right)$$

and accordingly the

Theorem 4.3. *The viscoelastic equilibrium is uniformly Liapunov stable with respect to the measures*

$$(4.8) \quad \mu(\mathbf{u}) = \|\dot{\mathbf{u}}(\cdot, t)\|_0^2 + \|\mathbf{u}(\cdot, t)\|_+^2 \quad \text{and} \quad \mu_0(\mathbf{u}) = \|\dot{\mathbf{u}}(\cdot, 0)\|_0^2 + \|\mathbf{u}(\cdot, 0)\|_+^2.$$

By using the Poincaré inequality

$$(4.9) \quad \|\mathbf{u}(\cdot, t)\|_0 \leq c \|\mathbf{u}(\cdot, t)\|_+, \quad t \in I, \quad c > 0,$$

from (4.7) we obtain the inequality

$$(4.10) \quad \|\dot{\mathbf{u}}(\cdot, t)\|_0^2 + \|\mathbf{u}(\cdot, t)\|_0^2 \leq \frac{k}{mm_1} \left(\|\dot{\mathbf{u}}(\cdot, 0)\|_0^2 + \|\mathbf{u}(\cdot, 0)\|_+^2 \right), \quad t \in I,$$

where $m_1 = \min \{1, 1/c_1\}$, and consequently we have the

Theorem 4.4. *The equilibrium viscoelastic solution is uniformly Liapunov stable with respect to the measures*

$$(4.11) \quad \nu(\mathbf{u}) = \|\dot{\mathbf{u}}(\cdot, t)\|_0^2 + \|\mathbf{u}(\cdot, t)\|_0^2 \quad \text{and} \quad \mathcal{E}_0(\mathbf{u}) = \tilde{E}(0),$$

and with respect to the measures $\nu(\mathbf{u})$ and $\mu_0(\mathbf{u})$.

From (2.4), (3.3), (3.5) and (3.6) we obtain

$$m \left(\|\dot{\mathbf{u}}(\cdot, t)\|_+^2 + a \|\mathbf{u}(\cdot, t)\|_+^2 \right) + b \int_0^t \|\mathbf{u}(\cdot, \tau)\|_+^2 d\tau \leq 2\tilde{E}(0), \quad t \in I,$$

and in consequence

$$(4.12) \quad ma \|\mathbf{u}(\cdot, t)\|_+^2 + b \int_0^t \|\mathbf{u}(\cdot, \tau)\|_+^2 d\tau \leq 2\tilde{E}(0), \quad t \in I.$$

This last inequality leads to

$$(4.13) \quad \int_0^t \|\mathbf{u}(\cdot, \tau)\|_+^2 d\tau \leq \frac{2}{b} (1 - e^{bt/ma}) \tilde{E}(0) \leq \frac{2}{b} \tilde{E}(0) \quad t \in I,$$

which together with (4.13) yields

$$(4.14) \quad \int_0^t \|\mathbf{u}(\cdot, \tau)\|_0^2 d\tau \leq \frac{2c^2}{b} (1 - e^{bt/ma}) \tilde{E}(0) \leq \frac{2c^2}{b} \tilde{E}(0) \quad t \in I.$$

Theorem 4.5. *If the elasticities C_{ijkl} are time-dependent the equilibrium viscoelastic solution is uniformly Liapunov stable with respect to the measures*

$$(4.15) \quad \varphi(\mathbf{u}) = \int_0^t \|\mathbf{u}(\cdot, \tau)\|_+^2 d\tau \quad \text{and} \quad \mathcal{E}_0(\mathbf{u}) = \tilde{E}(0),$$

and with respect to the measures

$$(4.16) \quad \psi(\mathbf{u}) = \int_0^t \|\mathbf{u}(\cdot, \tau)\|_0^2 d\tau \quad \text{and} \quad \mathcal{E}(\mathbf{u}) = \tilde{E}(0).$$

Remark 4.1. By making $t \rightarrow T \leq \infty$ in (4.13) and (4.14) we obtain the following estimations for the equilibrium solution of viscoelasticity:

$$(4.17) \quad \int_0^T \|\mathbf{u}(\cdot, \tau)\|_+^2 d\tau \leq \frac{2}{b} \tilde{E}(0), \quad \int_0^T \|\mathbf{u}(\cdot, \tau)\|_0^2 d\tau \leq \frac{2c^2}{b} \tilde{E}(0).$$

Remark 4.2. Guerriero [10] has proved that the equilibrium classical solution of viscoelasticity for the exterior of a region B is uniformly Liapunov stable with respect to the pairs of measures $\mathcal{E}_0(\mathbf{u})$, $\mathcal{E}(\mathbf{u})$, and $\mathcal{E}_0(\mathbf{u})$, $\varphi(\mathbf{u})$. We also point out the Beever's result [2] regarding the Hölder stability (see Def. 8.2 of [11]) of the null solutions of viscoelastic materials of convolutive type.

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REFERENCES

1. Adams R. A., *Sobolev Spaces*. Academic Press, New York, 1975.
2. Beever's C. E., *Hölder Stability in Linear Viscoelasticity*. "Mechanics of Visco-Elastic Media and Bodies", IUTAM Symposium, Göteborg, Sweden, 1974, pp. 190-195.
3. Ciobanu Gh. Gr., *Stability of the Solutions in Linear Viscoelasticity*. Bul. Inst. Polit. Iași. **XXIV (XXVIII)**, 1 - 2, s. I, 91-96 (1978).
4. Ciobanu Gh. Gr., *Sur la stabilité de l'équilibre viscoélastique*. Bull. Math. Soc. Sci. Math. de la R.S. de Roumanie, **24 (72)**, 3, 227-231 (1980).

5. Ciobanu Gh. Gr., *On Stability of Solution in Linear Viscoelasticity*. An. Șt. Univ. Iași, **XXX**, s. Ia, Matematică, 89-93 (1984).
6. Ciobanu Gh. Gr., *Stability and Continuous Dependence in Linear Viscoelasticity*. An. Șt. Univ. Iași, **XXXVII**, 3, s. Ia, Matematică, 299-314 (1991).
7. Dafermos C. M., *An Abstract Volterra Equation with Applications to Linear Viscoelasticity*. J. Diff. Equations, **7**, 554-569 (1970).
8. Dafermos C. M., *Asymptotic Stability in Viscoelasticity*. Arch. Rational Mech. Anal., **37**, 297-308 (1970).
9. Gurtin M. A., *The Linear Theory of Elasticity*. Handbuch der Physik, **VI a/2**, Springer Verlag, Berlin-Heidelberg-New York, pp. 1-295.
10. Guerriero G., *On the Stability Problem for the Exterior Linear Viscoelasticity*. Ricerche di Matematica, **XXXVI**, 1, 63-72 (1987).
11. Knops R. J., Wilkes E. W., *Theory of Elastic Stability*. Handbuch der Physik, **VI a/2**, Springer Verlag, Berlin-Heidelberg-New York, pp. 125-302.
12. Lions J. L., Magenes, *Problèmes aux limites non homogènes et applications*. Vol. I, Dunod, Paris, 1968.

ASUPRA STABILITĂȚII ECHILIBRULUI VASCOELASTIC

(Rezumat)

Se revine asupra unor rezultate ale autorului [3]-[6] privind stabilitatea echilibrului vâscoelastic în vederea unei mai bune precizări a acestor rezultate și pentru o prezentare unitară și concisă a acestora.

DYNAMICAL DISCONTINUITY RELATIONS IN THE LINEAR THEORY OF THERMOELASTICITY WITH MICROSTRUCTURE

BY

SILVIA-GABRIELA CIUMAȘU

Abstract. Following the method proposed by Chadwick and Powlis [1], a weak solution for the linear theory of thermoelasticity with microstructure is defined and the conditions of dynamical compatibility are established, as well as the basic equations for the study of shock waves propagation and conditions that their propagation speed be real and positive.

Key words: linear thermoelasticity, dynamical discontinuity, shock waves.

1. Introduction

In this paper we establish the dynamical discontinuity relations in the linear theory of thermoelastic material with microstructure following the method of Chadwick and Powlis [1].

The propagation conditions of shock waves are given.

Propagation of shock waves in the classical linear theory of elastic bodies with microstructure was considered in [7].

2. Basic Equations

Let B be a closed region in E_3 which is occupied by a homogeneous and anisotropic elastic material with microstructure. We denote by I the time interval during which the body is disturbed from its reference state.

The basic equations in the linear thermoelasticity with microstructure, in the absence of the body forces and the externally supplied specific heat, are [2], [5]:

i) the *equation of motion*

$$(2.1) \quad t_{ij,i} = \rho \dot{v}_j; \quad \sigma_{ij} + \mu_{kij,k} = \rho I_{is} \dot{v}_{sj},$$

where $v_i = \partial u_i / \partial t$, $v_{ij} = \partial \varphi_{ij} / \partial t$;

ii) the equation of energy

$$(2.2) \quad \rho T_0 \dot{\eta} - q_{i,i} = 0;$$

iii) the constitutive equations

$$(2.3) \quad \begin{aligned} \tau_{ij} &= c_{ijkl} \varepsilon_{kl} + g_{klij} \gamma_{kl} + f_{krsij} \kappa_{krs} - a_{ij} \theta, \\ \sigma_{ij} &= g_{ijkl} \varepsilon_{kl} + b_{ijkl} \gamma_{kl} + d_{ijrls} \kappa_{rls} - c_{ij} \theta, \\ \mu_{kij} &= f_{kijls} \varepsilon_{ls} + d_{ls kij} \gamma_{ls} + a_{kijlsm} \kappa_{lsm} - a_{kij} \theta, \\ \rho \eta &= a \theta + a_{ij} \varepsilon_{ij} + c_{ij} \gamma_{ij} + a_{kij} \kappa_{kij}, \\ q_i &= k_{ij} \theta_{,j}; \end{aligned}$$

iv) the geometrical equations

$$(2.4) \quad 2\varepsilon_{ij} = u_{i,j} + u_{j,i}; \quad \gamma_{ij} = u_{j,i} - \varphi_{ij}; \quad \kappa_{ijk} = \varphi_{jk,i}.$$

In the above relations we have used the following notations: $t_{ij} = \tau_{ij} + \sigma_{ij}$; τ_{ij} – the classical stress tensor; σ_{ij} – the relative stress tensor; μ_{ijk} – the couple stress tensor; ε_{ij} – the classical infinitesimal strain; γ_{ij} – the relative deformation tensor; κ_{ijk} – the microdeformation gradient tensor; q_i – the components of the heat flux vector; η – the specific entropy; u_i – the displacement vector; φ_{ij} – the microdeformation tensor; θ – the temperature measured from a constant reference temperature T_0 ; c_{ijkl} , b_{ijkl} , ..., a_{kij} – the coefficients of material; I_{ij} – coefficients of inertia of micro-material.

The coefficients satisfy the symmetry relations:

$$(2.5) \quad \begin{aligned} c_{ijkl} &= c_{klij} = c_{jikl}; & g_{ijkl} &= g_{ijlk}; & b_{ijkl} &= b_{klij}; & a_{ij} &= a_{ji}; \\ a_{ijklsm} &= a_{lsmijk}; & I_{ij} &= I_{ji}; & k_{ij} &= k_{ji}; & k_{ij} \theta_{,i} \theta_{,j} &\geq 0. \end{aligned}$$

If the functions $(u_i, \varphi_{ij}, \theta)$ are of class C^2 on $\mathcal{B} = B \times I$ and satisfy the equations (1.1)–(1.4), then $(u_i, \varphi_{ij}, \theta)$ is a strict solution of these equation on $\mathcal{B}$.

3. Weak Solution of the Linear Thermoelasticity with Microstructure Equations

We denote $\mathcal{B} = B \times I$ and let $D_4(\mathcal{B})$ be the set of all regular regions contained in $\mathcal{B}$; if $D \in D_4(\mathcal{B})$ we denote by D^0 the interior and by ∂D the boundary of D .

Following [1] in order to define the weak solutions we introduce a class of test functions. Let $\mathcal{L}$ be an open region in $\mathcal{E} = \mathbb{R}^3 \times I$ – the set of ordered pairs $(t, \mathbf{x})$ where $t \in I$, $\mathbf{x} \in \mathbb{R}^3$.

Definition. A function $\varphi = (\xi_i, \xi_{jk}, \xi)$, $\varphi : \mathcal{E} \rightarrow R^{13}$ will be called a test function on $\mathcal{L}$ if it is of class C^∞ on $\mathcal{E}$ and is zero in $\mathcal{E} \setminus \mathcal{L}$.

Definition. If the $(u_i, \varphi_{ij}, \theta)$ are related on $\mathcal{B}$ to auxiliary variables $v_i, v_{ij}, t_{ij}, \mu_{kij}, \eta, q_i$, by Eqs. (1.3), (1.4) and for all $D \in D_4(\mathcal{B})$ and for all sets (ξ_i, ξ_{jk}, ξ) of test functions on D^0 , the auxiliary variables satisfy the equations:

$$(3.1) \quad \begin{aligned} \int_D \left(\rho v_p \frac{\partial \xi_p}{\partial t} - t_{qp} \frac{\partial \xi_p}{\partial x_q} \right) d\tau &= 0, \\ \int_D \left(\rho I_{is} v_{sj} \frac{\partial \xi_{ij}}{\partial t} - \mu_{kij} \frac{\partial \xi_{ij}}{\partial x_k} + \sigma_{ij} \xi_{ij} \right) d\tau &= 0, \\ \int_D \left(\rho T_0 \eta \frac{\partial \xi}{\partial t} - q_i \frac{\partial \xi}{\partial x_i} \right) d\tau &= 0; \end{aligned}$$

we refer to $(u_i, \varphi_{ij}, \theta)$ as a weak solution of the linear thermoelasticity with microstructure on $\mathcal{B}$.

If $(u_i, \varphi_{ij}, \theta)$ is a weak solution of Eqs. (2.1), (2.2) on $\mathcal{B}$ and the functions are of class C^2 on $\mathcal{B}$, then $(u_i, \varphi_{ij}, \theta)$ is a strict solution of these equations on $\mathcal{B}$. We have

$$(3.2) \quad \begin{aligned} \int_D \xi_i \left(\rho \frac{\partial v_i}{\partial t} - \frac{\partial t_{ji}}{\partial x_j} \right) d\tau &= \int_D \left[\frac{\partial}{\partial t} (\rho v_i \xi_i) - \frac{\partial}{\partial x_j} (t_{ji} \xi_i) \right] d\tau = \\ &= \int_{\partial D} \xi_i (\rho v_i N_0 - t_{ji} N_j) d\sigma = 0, \\ \int_D \xi_{ij} \left(\rho I_{is} \frac{\partial v_{sj}}{\partial t} - \frac{\partial \mu_{kij}}{\partial x_k} - \sigma_{ij} \right) d\tau &= \int_D \left[\frac{\partial}{\partial t} (\rho I_{is} v_{sj} \xi_{ij}) - \frac{\partial}{\partial x_k} (\mu_{kij} \xi_{ij}) \right] d\tau = \\ &= \int_{\partial D} \xi_{ij} (\rho I_{is} v_{ij} N_0 - \mu_{kij} N_k) d\sigma = 0, \\ \int_D \xi \left(\rho T_0 \frac{\partial \eta}{\partial t} - \frac{\partial q_i}{\partial x_i} \right) d\tau &= \int_D \left[\frac{\partial}{\partial t} (\rho T_0 \eta \xi) - \frac{\partial}{\partial x_i} (\xi q_i) \right] d\tau = \\ &= \int_{\partial D} \xi (\rho T_0 \eta N_0 - q_i N_i) d\sigma = 0, \end{aligned}$$

where use has been made of the divergence theorem, (N_0, N_1, N_2, N_3) being the direction cosines of a unit normal vector on ∂D . Since (ξ_i, ξ_{jk}, ξ) vanish on ∂D the first member of the above equations is zero and since the integrand is continuous on B we recover the equation of motion (2.1), (2.2).

4. Dynamical Discontinuity Relations

Let $(u_i, \varphi_{ij}, \theta)$ be a weak solution of the field equations on $\mathcal{B}$, which contains a singular surface Σ . Let V be an open region of $\mathcal{B}$ such that $\Omega = V \times I$ intersects Σ . We assume that $(u_i, \varphi_{ij}, \theta)$ are of class C^2 on V^+ and on V^- .

Let $D \subset V$ be an arbitrary regular region which intersects Σ , the sets $D^+ = D \cap V^+$ and $D^- = D \cap V^-$ both being non-empty. Since $(u_i, \varphi_{ij}, \theta)$ is a strict solution of the field equations on V^+ and on V^- , Eqs. (2.1), (2.2) hold on D^+ and D^- .

Then we can write:

$$(4.1) \quad \begin{aligned} \int_{D^+ \cup D^-} \left[\frac{\partial}{\partial t} (\rho v_i \xi_i) - \frac{\partial}{\partial x_j} (t_{ji} \xi_i) \right] d\tau &= 0, \\ \int_{D^+ \cup D^-} \left[\frac{\partial}{\partial t} (\rho I_{is} v_{sj} \xi_{ij}) - \frac{\partial}{\partial x_j} (\mu_{kij} \xi_{ij}) \right] d\tau &= 0, \\ \int_{D^+ \cup D^-} \left[\frac{\partial}{\partial t} (\rho T_0 \eta \xi) - \frac{\partial}{\partial x_i} (\xi q_i) \right] d\tau &= 0. \end{aligned}$$

We have for the direction cosines of a unit normal vector on ∂D the relation [4]

$$N_0 = -AG, \quad N_i = An_i \quad \text{on} \quad \Sigma,$$

where A is a normalizing function, G is the speed of surface Σ in the direction of the unit normal vector whose components are n_i .

By using the divergence theorem and the fact that the test functions vanish and ∂D , we obtain:

$$(4.2) \quad \begin{aligned} \int_{\Sigma \cap D} A \xi_i \{ \rho G [v_i] + [t_{ji}] n_j \} d\sigma &= 0, \\ \int_{\Sigma \cap D} A \xi_{ij} \{ \rho I_{is} G [v_{sj}] + [\mu_{kij}] n_k \} d\sigma &= 0, \\ \int_{\Sigma \cap D} A \xi \{ \rho T_0 G [\eta] + [q_i] n_i \} d\sigma &= 0. \end{aligned}$$

In view of the continuity of the integrands on Σ , from (4.2) we obtain the dynamical discontinuity relations:

$$\begin{aligned} (4.3) \quad & \rho G [v_i] + [t_{ji}] n_j = 0, \\ & \rho I_{is} G [v_{sj}] + [\mu_{kij}] n_k = 0, \\ & \rho T_0 G [\eta] + [q_i] n_i = 0 \quad \text{on} \quad \Sigma. \end{aligned}$$

5. Propagation of Shock Waves

A propagation surface $\Sigma(t)$ is said to be a shock wave if the function u_i , φ_{ij} , θ are continuous everywhere, while their first order derivatives have jump discontinuities across $\Sigma(t)$ but are continuous functions everywhere else.

In jump of a function across $\Sigma(t)$ is denoted by $[Z] = Z^- - Z^+$, where Z^+ is the value of Z immediately ahead of the wavefront and Z^- is the value of Z immediately behind it.

The conditions of compatibility for shock waves are [4, §39], [7]:

$$\begin{aligned} (5.1) \quad & [u_{i,j}] = \lambda_i n_j; \quad [\dot{u}_i] = -G \lambda_i; \quad [\varphi_{ij,k}] = \lambda_{ij} n_k; \quad [\dot{\varphi}_{ij}] = -G \lambda_{ij}; \\ & \lambda_i = \left[\frac{\partial u_i}{\partial t} \right]; \quad \lambda_{ij} = \left[\frac{\partial \varphi_{ij}}{\partial x_k} \right]; \quad \left[\frac{\partial \theta}{\partial n} \right] = \psi, \end{aligned}$$

where G is the speed of propagation of the wave, n_i are the components of the unit normal of $\Sigma(t)$.

Taking into account the relations (2.3), (2.4) and (5.1), the jump conditions (4.3), can be written as

$$\begin{aligned} (5.2) \quad & A_{qj} \lambda_j + B_{qjk} \lambda_{jk} = \rho G^2 \lambda_q, \\ & B_{qrj} \lambda_j + C_{qrjk} \lambda_{jk} = \rho G^2 \sigma_{qrjk} \lambda_{jk}, \\ (5.3) \quad & \rho T_0 G (d_i \lambda_i + C_{kj} \lambda_{jk}) + k_{ij} \psi = 0, \end{aligned}$$

where

$$\begin{aligned} (5.4) \quad & A_{qj} = A_{ijqp} n_i n_p = (c_{pqji} + g_{pqji} + g_{ijpq} + b_{ijpq}) n_i n_p, \\ & B_{qjk} = B_{pqijk} n_i n_p = (f_{ijkpq} + d_{pqijk}) n_i n_p, \\ & C_{qrjk} = a_{pqrijk} n_i n_p, \quad \sigma_{qrjk} = I_{qj} \delta_{kr}, \\ & d_i = a_{ij} n_j + c_{ij} n_i, \quad C_{kj} = a_{kij} n_i. \end{aligned}$$

We may write the Eqs. (5.2) in the form

$$(5.5) \quad (Q - \rho G^2 E) \Lambda = 0,$$

where $Q = \beta^{-1} \alpha$,

$$\alpha = (\alpha_{ij})_{12 \times 12} = \begin{pmatrix} \tilde{a} & \tilde{b}^T \\ \tilde{b} & \tilde{c} \end{pmatrix}, \quad \beta = (\beta_{ij})_{12 \times 12} = \begin{pmatrix} \tilde{E} & \tilde{O}^T \\ \tilde{O} & \tilde{S} \end{pmatrix},$$

$$\tilde{a} = (A_{ip})_{3 \times 3}, \quad \tilde{b} = (B_{qjk})_{9 \times 3}, \quad \tilde{E} = (\delta_{ij})_{3 \times 3}, \quad \Lambda = (\lambda_1 \quad \lambda_2 \quad \dots \quad \lambda_{12})^T,$$

$$\tilde{c} = (C_{qrjk})_{9 \times 9}, \quad \tilde{O} = \begin{pmatrix} O = 0 \\ \tilde{O} \end{pmatrix}_{9 \times 3}, \quad \tilde{S} = (\sigma_{qrjk})_{9 \times 9},$$

$$(jk) \Rightarrow M \in (4, 5, \dots, 12), \quad (qr) \rightarrow N \in (4, 5, \dots, 12).$$

Theorem. *If the coefficients satisfy the symmetry relations (5.4) and we have*

$$(5.6) \quad \sigma_{jkqr} \psi_{jk} \psi_{qr} \geq \gamma \psi_{jk} \psi_{jk}, \quad \forall \psi_{jk}, \quad \gamma > 0;$$

$$\frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + g_{ijkl} \varepsilon_{kl} \gamma_{ij} + \frac{1}{2} b_{ijkl} \gamma_{ij} \gamma_{kl} + f_{kijls} \kappa_{kij} \varepsilon_{ls} +$$

$$(5.7) \quad + d_{kijls} \kappa_{jls} \gamma_{ki} + \frac{1}{2} a_{kijlsm} \kappa_{kij} \kappa_{lsm} \geq 0$$

$$(\varepsilon_{ij} \neq 0, \quad \gamma_{ij} \neq 0, \quad \kappa_{kij} \neq 0),$$

then results that all the speed of propagation of Σ are real and positive.

For $\Sigma(t)$ to be a singular surface, the vector Λ cannot vanish identically. The Eqs. (5.2), (5.3) has a not trivial solution for (Λ, ψ) if only if

$$(5.7) \quad k_{ij} \det (Q - \rho G^2 E) = 0.$$

The vector Λ is an eigenvector of Q , while the speed of propagation G is the corresponding eigenvalue. From (5.5), (5.6) it follows that all the roots of the Eq. (5.7) are real and positive.

By using the jump operator in Eqs. (2.1), (2.2) as well as the compatibility conditions for $[\theta_{,ii}]$, $[u_{i,jk}]$, $[\varphi_{jk,ip}]$, $[\ddot{u}_i]$, $[\ddot{\varphi}_{ij}]$ ([7], [4, §39]) we obtain a system of equations [7], [4], which together with (5.2), (5.3) are the basic equations for the study of propagation of shock waves in an anisotropic homogeneous linearly thermoelastic material with microstructure.

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REFERENCES

1. Chadwick P., Powdrill B., *Singular Surface in Linear Thermoelasticity*. Int. J. Eng. Sci., **3**, 561-595 (1965).

2. Green A. E., Rivlin S., *Simple Forces and Stress Multipolar*. Arch. Rat. Mech. Anal., **16**, 325-353 (1964).
3. * * * *Handbuch der Physik*. Band III/1, Springer Verlag, Berlin-Göttingen-Heidelberg, 1960.
4. Ieșan D., *Teoria termoelasticității*. Ed. Acad. R.S.R., București, 1979.
5. Ieșan D., *Mecanica generalizată a solidelor*. Univ. "Al. I. Cuza", Iași, 1980.
6. Mindlin R., *Microstructure in Linear Elasticity*. Arch. Rat. Mech. and Anal., **16**, 51-78 (1964).
7. Neagu S. G., *Wave Propagation in an Anisotropic Homogeneous Linearly Elastic Material with Microstructure*. Bul. Inst. Polit. Iași, **XXV(XXIX)**, 3-4, s. IV, 25-30 (1979).

RELAȚII DE DISCONTINUITATE DINAMICĂ ÎN TEORIA TERMOELASTICITĂȚII CU MICROSTRUCTURĂ

(Rezumat)

Folosind metoda lui P. Chadwick și B. Poldrill [1], se definește o soluție slabă pentru ecuațiile teoriei termoelasticității cu microstructură [2], [5]. Se deduc condițiile de compatibilitate dinamice în această teorie și ecuațiile care stau la baza studiului hipersuprafețelor singulare de ordinul întâi. Se dau condițiile pentru ca vitezele de propagare ale unde de șoc să fie reale și pozitive, într-un material termoelastic omogen, anizotrop cu microstructură.

1. The first part of the report is devoted to a general survey of the situation in the country. It is followed by a detailed account of the work done during the year. The third part contains a list of the names of the persons who have been engaged in the work, and a list of the names of the persons who have been engaged in the work during the year. The fourth part contains a list of the names of the persons who have been engaged in the work during the year.

REPORT OF THE DIRECTOR OF THE BUREAU OF AGRICULTURE

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1. Introduction

The first part of the report is devoted to a general survey of the situation in the country. It is followed by a detailed account of the work done during the year. The third part contains a list of the names of the persons who have been engaged in the work, and a list of the names of the persons who have been engaged in the work during the year. The fourth part contains a list of the names of the persons who have been engaged in the work during the year.

2. Review of the Year's Work

The first part of the report is devoted to a general survey of the situation in the country. It is followed by a detailed account of the work done during the year. The third part contains a list of the names of the persons who have been engaged in the work, and a list of the names of the persons who have been engaged in the work during the year. The fourth part contains a list of the names of the persons who have been engaged in the work during the year.

FINITE ELEMENT FOR LINEAR THEORY OF MICROPOLAR ELASTICITY

BY

D. MANOLE

Abstract. The aim of this paper is to give finite element method for linear theory of micropolar elasticity.

Key words: discretization, elasticity, finite element, micropolar, potential.

1. Introduction

Finite element of elastics bodies has been presented in several papers [4]...[7]. Finite element analysis for the case a micropolar elastic material are given in [8].

It is the aim of the present paper to establish a finite element for linear theory of micropolar elasticity.

2. Finite Element Approximations

Throughout this paper, we employ a rectangular coordinate system x_k and the indicial notation. We consider a regular region Ω of space occupied by a micropolar elastic material, whose Lipschitz boundary $\partial\Omega$. The basic equations in the linear theory of these bodies are [1]:

i) the *equilibrium equations*

$$(2.1) \quad t_{ji,j} + f_i = 0, \quad m_{ji,j} + \varepsilon_{ijk} t_{jk} + l_i = 0;$$

ii) the *constitutive equations*

$$(2.2) \quad t_{ij} = A_{ijkl} \gamma_{kl} + B_{ijkl} \chi_{kl}, \quad m_{ij} = B_{klij} \gamma_{kl} + C_{ijkl} \chi_{kl};$$

iii) the *geometrical equations*

$$(2.3) \quad \gamma_{ij} = u_{j,i} + \varepsilon_{jik} \varphi_k, \quad \chi_{ij} = \varphi_{ji}, \quad \text{in } \Omega,$$

where

$$(2.4) \quad A_{ijkl} = A_{klij}, \quad C_{ijkl} = C_{klij}.$$

To the system of field equations, we add boundary conditions:

$$(2.5) \quad \begin{aligned} u_i &= \tilde{u}_i \quad \text{on } S_1, & \varphi_i &= \tilde{\varphi}_i \quad \text{on } S_3, \\ t_i &= t_{ij}n_j = \tilde{t}_i \quad \text{on } S_2, & m_i &= m_{ij}n_j = \tilde{m}_i \quad \text{on } S_4, \end{aligned}$$

where S_r ($r = 1, 2, 3, 4$) are parts of $\partial\Omega$ such that

$$S_1 \cup S_2 = S_3 \cup S_4 = \partial\Omega, \quad S_1 \cap S_2 = S_3 \cap S_4 = \phi.$$

In the above equations, we have used the following notation: t_{ij} are components of the stress tensor; m_{ij} are components of the couple stress tensor; u_i are components of the displacement vector; φ_i are components of the microrotation vector; γ_{ij} , χ_{ij} are components of deformation; ε_{ijk} is alternating symbol; f_i are components of body force; l_i are components of the couple; n_i are components of the unit outward normal to $\partial\Omega$; the comma denotes partial derivation with respect to the variable x_i ; A_{ijkl} , B_{ijkl} , C_{ijkl} are functions characterizing the respective material and $\tilde{u}_i$, $\tilde{\varphi}_i$, $\tilde{t}_i$, $\tilde{m}_i$ are prescribed functions.

The first step in a finite element approximation is to derive a weak form of the boundary value problem (2.1)–(2.5). Let $\mathbf{u} = (u_1, u_2, u_3, \varphi_1, \varphi_2, \varphi_3)$, $\mathbf{v} = (v_1, v_2, v_3, \omega_1, \omega_2, \omega_3)$ be and $\bar{\mathbf{u}} = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)$ the virtual displacement-microrotation vector. We have $\bar{u}_i = 0$ on S_1 and $\bar{\varphi}_i = 0$ on S_3 .

Multiplying (2.1)₁ by $\bar{u}_i$ and (2.1)₂ by $\bar{\varphi}_i$, summing up, integrating over Ω , applying the divergence theorem and taking into account (2.3), it results that

$$(2.6) \quad \begin{aligned} & \int_{\Omega} (t_{ij}(u)\gamma_{ij}(\bar{\mathbf{u}}) + m_{ij}(\mathbf{u})\chi_{ij}(\bar{\mathbf{u}})) d\Omega = \\ & = \int_{\Omega} (f_i\bar{u}_i + l_i\bar{\varphi}_i) d\Omega + \int_S (t_{ij}\bar{u}_i n_j + m_{ij}\bar{\varphi}_i n_j) d\sigma \end{aligned}$$

Applying the boundary conditions (2.5), we obtain the weak form:

$$(2.7) \quad \begin{aligned} & u_i = \tilde{u}_i \quad \text{on } S_1, \quad \varphi_i = \tilde{\varphi}_i \quad \text{on } S_3, \\ & \int_{\Omega} (t_{ij}(\mathbf{u})\gamma_{ij}(\bar{\mathbf{u}}) + m_{ij}(\mathbf{u})\chi_{ij}(\bar{\mathbf{u}})) d\Omega = \\ & = \int_{\Omega} (f_i\bar{u}_i + l_i\bar{\varphi}_i) d\Omega + \int_{S_2} \tilde{t}_i\bar{u}_i d\sigma + \int_{S_4} \tilde{m}_i\bar{\varphi}_i d\sigma, \end{aligned}$$

$$\forall \mathbf{u} \ni \bar{u}_i = 0 \quad \text{on } S_1 \quad \text{and} \quad \bar{\varphi}_i = 0 \quad \text{on } S_3.$$

On the basis of (2.4), the weak form (2.7) could be derived from the principle of minimum potential energy (see [2]).

The principle of minimum potential energy then says that the field u corresponding to the equilibrium configuration minimizes the total potential energy F among all possible displacements that satisfy the condition $u_i = \tilde{u}_i$ on S_1 and all possible microrotations satisfy the condition $\varphi_i = \tilde{\varphi}_i$ on S_1 and all possible microrotations satisfy the condition $\varphi_i = \tilde{\varphi}_i$ on S_3 , that is

$$(2.8) \quad \begin{aligned} u_i = \tilde{u}_i \quad \text{on} \quad S_1, \quad \varphi_i = \tilde{\varphi}_i \quad \text{on} \quad S_3 \quad \text{and} \quad F(u) \leq F(v), \\ \forall v \ni v_i = \tilde{u}_i \quad \text{on} \quad S_1, \quad \omega_i = \tilde{\varphi}_i \quad \text{on} \quad S_3, \quad (\text{see (3.13) [8], [3]}). \end{aligned}$$

If, in (2.8), we take $v = u \pm \theta \bar{u}$ with $\theta \in (0, 1]$ and $\bar{u}$ an arbitrary displacement-microrotation field such that $\bar{u}_i = 0$ on S_1 , $\bar{\varphi}_i = 0$ on S_3 , we have

$$(2.9) \quad \begin{aligned} & \pm \theta \left\{ \int_{\Omega} (t_{ij}(u) \gamma_{ij}(\bar{u}) + m_{ij}(u) \chi_{ij}(\bar{u})) d\Omega \right\} \mp \\ & \mp \theta \left\{ \int_{\Omega} (f_i \bar{u}_i + l_i \bar{\varphi}_i) d\Omega + \int_{S_2} \tilde{t}_i \bar{u}_i d\sigma + \int_{S_4} \tilde{m}_i \bar{\varphi}_i d\sigma \right\} + \\ & + \frac{1}{2} \theta^2 \left\{ \int_{\Omega} (t_{ij}(\bar{u}) \gamma_{ij}(\bar{u}) + m_{ij}(\bar{u}) \chi_{ij}(\bar{u})) d\Omega \right\} \geq 0. \end{aligned}$$

Dividing by $\pm \theta$ and letting $\theta \rightarrow 0$, we obtain the form

$$\begin{aligned} & \int_{\Omega} (t_{ij}(u) \gamma_{ij}(\bar{u}) + m_{ij}(u) \chi_{ij}(\bar{u})) d\Omega = \\ & = \int_{\Omega} (f_i \bar{u}_i + l_i \bar{\varphi}_i) d\Omega + \int_{S_2} \tilde{t}_i \bar{u}_i d\sigma + \int_{S_4} \tilde{m}_i \bar{\varphi}_i d\sigma, \end{aligned}$$

$$\forall u \ni \bar{u}_i = 0, \quad \text{on} \quad S_1 \quad \text{and} \quad \bar{\varphi}_i = 0 \quad \text{on} \quad S_3.$$

This is same as (2.7).

The next step is to discretize the domain Ω into a set of finite elements and to define the shape functions $\{N_\alpha(x)\}$ and $\{M_\alpha(x)\}$ on each finite element Ω_e . Then the displacement-microrotation u and the virtual displacement-microrotation $\bar{u}$ are approximated by linear combinations of their nodal values and their shape functions:

$$(2.10) \quad u_j = u_{j\beta} N_\beta(x), \quad \varphi_i = \varphi_{j\beta} M_\beta(x), \quad \bar{u}_i = \bar{u}_{i\alpha} N_\alpha(x), \quad \bar{\varphi}_i = \bar{\varphi}_{i\alpha} M_\alpha(x),$$

in each finite element Ω_e .

Using relations (2.10) from (2.7), we have discretization of the weak form (2.7):

$$(2.11) \quad \sum_{e=1}^E \left(\bar{u}_{j\beta} A_{j\beta l\alpha}^e u_{l\alpha} + \bar{u}_{j\beta} B_{j\beta l\alpha}^e \varphi_{l\alpha} + \bar{\varphi}_{j\beta} C_{l\alpha j\beta}^e u_{l\alpha} + \bar{\varphi}_{j\beta} D_{l\alpha j\beta}^e \varphi_{l\alpha} \right) =$$

$$= \sum_{e=1}^E \left(\bar{u}_{j\beta} f_{j\beta}^{1,e} + \bar{\varphi}_{j\beta} l_{j\beta}^{1,e} \right) + \sum_{e=1}^{E_2} \bar{u}_{j\beta} f_{j\beta}^{2,e} + \sum_{e=1}^{E_4} \bar{\varphi}_{j\beta} f_{j\beta}^{4,e},$$

$$\forall \bar{u}_{j\beta} \ni \bar{u}_{j\beta} = 0 \quad \text{on } S_1, \quad \forall \bar{\varphi}_{j\beta} \ni \bar{\varphi}_{j\beta} = 0 \quad \text{on } S_3,$$

where

$$(2.12) \quad A_{j\beta l\alpha}^e = \int_{\Omega_e} A_{ijkl} N_{\alpha,k} N_{\beta,i} d\Omega,$$

$$B_{j\beta l\alpha}^e = \int_{\Omega_e} (B_{ijkl} N_{\beta,i} M_{\alpha,k} + A_{ijkl} \varepsilon_{rkl} N_{\beta,i} M_{\alpha}) d\Omega,$$

$$C_{j\beta l\alpha}^e = \int_{\Omega_e} (B_{ijkl} N_{\beta,i} M_{\alpha,k} + A_{irkl} \varepsilon_{ril} N_{\beta,k} M_{\alpha}) d\Omega,$$

$$D_{j\beta l\alpha}^e = \int_{\Omega_e} (B_{irkl} \varepsilon_{rij} M_{\beta} M_{\alpha,k} + C_{kl ij} M_{\beta,i} M_{\alpha,k} +$$

$$+ A_{ikr} \varepsilon_{rkj} \varepsilon_{qil} M_{\beta} M_{\alpha} + B_{irkj} \varepsilon_{ril} M_{\beta,k} M_{\alpha}) d\Omega,$$

$$f_{i\alpha}^{1,e} = \int_{\Omega_e} f_i N_{\alpha} d\Omega, \quad l_{i\alpha}^{1,e} = \int_{\Omega_e} l_i M_{\alpha} d\Omega,$$

$$f_{i\alpha}^{2,e} = \int_{S_2^e} \tilde{t}_i N_{\alpha} d\sigma, \quad f_{i\alpha}^{4,e} = \int_{S_4^e} \tilde{m}_i M_{\alpha} d\sigma.$$

Since the virtual displacement-microrotation is arbitrary, (2.11) yields the finite element equation

$$(2.13) \quad u_{i\alpha} = \bar{u}_{i\alpha} \quad \text{on } S_1, \quad \varphi_{i\alpha} = \bar{\varphi}_{i\alpha} \quad \text{on } S_3,$$

$$\sum_{e=1}^E \left(A_{j\beta l\alpha}^e u_{l\alpha} + B_{j\beta l\alpha}^e \varphi_{l\alpha} + C_{l\alpha j\beta}^e u_{l\alpha} + D_{l\alpha j\beta}^e \varphi_{l\alpha} \right) =$$

$$= \sum_{e=1}^E \left(f_{j\beta}^{1,e} + l_{j\beta}^{1,e} \right) + \sum_{e=1}^{E_2} f_{j\beta}^{2,e} + \sum_{e=1}^{E_4} f_{j\beta}^{4,e}.$$

Introducing the matrix representation of the strain-displacement-microrotation relation

$$(2.14) \quad \eta = \begin{pmatrix} \partial_{,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial_{,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_{,3} & 0 & 0 & 0 \\ \partial_{,2} + \varphi_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial_{,3} + \varphi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_{,1} + \varphi_2 & 0 & 0 & 0 \\ \partial_{,3} - \varphi_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial_{,1} - \varphi_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_{,2} - \varphi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_{,1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_{,2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_{,3} \\ 0 & 0 & 0 & \partial_{,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_{,3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_{,1} \\ 0 & 0 & 0 & \partial_{,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_{,1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_{,2} \end{pmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix},$$

where

$$\eta^T = (\gamma_{11} \ \gamma_{22} \ \gamma_{33} \ \gamma_{21} \ \gamma_{32} \ \gamma_{13} \ \gamma_{31} \ \gamma_{12} \ \gamma_{23} \ \chi_{11} \ \chi_{22} \ , \ \chi_{33} \ \chi_{21} \ \chi_{32} \ \chi_{13} \ \chi_{31} \ \chi_{12} \ \chi_{23}),$$

and $\partial_{,i} = \frac{\partial}{\partial x_i}$, we have

$$(2.15) \quad \int_{\Omega_e} (t_{ij}(\mathbf{u})\gamma(\bar{\mathbf{u}}) + m_{ij}(\mathbf{u})\chi_{ij}(\bar{\mathbf{u}})) d\Omega = \bar{w}_p \int_{\Omega_e} R_{Ip} P_{IJ} R_{Jq} d\Omega w_q.$$

In (2.15), we have:

$$\{w_q\} = \{u_{11} \ u_{21} \ u_{31} \ \varphi_{11} \ \varphi_{21} \ \varphi_{31} \ u_{12} \ u_{22} \ u_{32} \dots \varphi_{3\alpha}\},$$

where $u_{1\alpha}$, $\varphi_{1\alpha}$ are the x components of $\mathbf{u}$ at the α th node, α is the total number of nodes in an element,

$$P = \begin{bmatrix} A & B \\ B & C \end{bmatrix},$$

A, B, C is the matrix of components $A_{ijkl}(x), B_{ijkl}(x), C_{ijkl}(x)$ and R is the matrix

Here $p = 6(\alpha - 1) + i$, $q = 6(\beta - 1) + j$ for four-node quadrilateral element. Thus, the stiffness matrix is obtained by

$$(2.17) \quad K^e = \int_{\Omega_e} R^T P R d\Omega,$$

that is, in component form

$$(2.18) \quad K_{pq}^e = \int_{\Omega_e} R_{Ip} P_{IJ} R_{Jq} d\Omega.$$

The next procedure after obtaining the element stiffness matrix (2.16) and the load vector (see [2]) is assembling to the global stiffness matrix and the global vector (see [2]).

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REFERENCES

1. Ieșan D., *Unele probleme fundamentale ale teoriei liniare a mediilor elastice de tip Cosserat*. "Probleme actuale in mecanica solidelor", Vol. I, Ed. Academiei, București, 1975.
2. Kikuchi N., *Finite Element Methods in Mechanics*. Cambridge University Press, 1986.
3. Krizek M., Neittaanmaki P., *Finite Element Approximation of Variational Problems and Applications*. Longman Scientific & Technical, New York, 1990.
4. Hlaváček I., Lovisek J., *A Finite Element Analysis for the Signorini Problem in Plane Elastostatics*. Appl. Mat., **22**, 215-228 (1977).
5. Haslinger J., Hlaváček I., *A Mixed Finite Element Method Close to the Equilibrium Model Applied to Plane Elastostatics*. Appl. Mat., **21**, 28-42 (1976).
6. Haslinger J., Hlaváček I., *Contact Between Bodies*. (I) *Continuous Problems*. Appl. Mat., **25**, 324-347 (1980); (II) *Finite Element Analysis*. Appl. Mat., **26**, 263-290 (1981); (III) *Dual Finite Element Analysis*. Appl. Mat., **26**, 321-345 (1981).
7. Krizek M., *An Equilibrium Finite Element Method in Tree-dimensional Elasticity*. Appl. Mat., **27**, 46-75 (1982).
8. Manole D., *Finite Element Analysis for the Case of a Micropolar Elastic Material (I)* (to be published).

ELEMENT FINIT PENTRU TEORIA LINIARĂ A ELASTICITĂȚII MICROPOLARE

(Rezumat)

Punctul de plecare in metoda elementului finit din teoria liniară a elasticității micropolare este modelul integral (2.7). Acest model a fost obținut, in cadrul acestei teorii, din ecuațiile (2.1)–(2.5). După ce se obține matricea de rigiditate pe un domeniu elemental Ω_e , rezolvarea sistemului de ecuații obținut după asamblare se poate face prin diverse metode iterative.

STUDY OF THE STATE OF STRESS OF ELASTIC TUBE FOR BIG TORSION ANGLES

BY

MIRCEA LUPU and D. NICOARĂ

Abstract. Torsion of bars has always been an important topic in the mechanics of solids. This paper treats the mathematical model of the torsion problem for cylindrical elastic bars with a big torsion angle. We deduce the stress and we do a comparative study of the torsion problem for a tube and for a cylindrical bar with fixed or free ends.

Key words: torsion angle, state of stress, elastic tube.

1. Fundamental Equations

Consider an isotropic homogeneous cylindrical elastic bar. The lateral surface is free from external forces; body forces are absent. The region of section is assumed to be finite (simply connected or multiply connected). Let the origin of coordinates be taken at the centroid of the end section $B_1(z = 0)$ with the z axis directed parallel to the generators (along the geometrical axis of the bar), and the x and y axis being directed arbitrarily. The end B_2 is obtained for $z = l$, where l is the length of the bar (sufficiently long). The ends are acted on by distributed forces reducing to twisting moments M of opposite sense $M_3^0(B_2) = -M_3^0(B_1) = Mk$. This is the classical model of the torsion [2], [3], [5].

In the present paper we investigate the pure torsion of an isotropic hollow bar (tube) of radii a and b , whose material have a big torsion angle. To obtain the equations of elastic equilibrium we adopt the I s l i n s k i's mathematical model of the torsion [1].

Consider now, an element of length l situated at a distance r of the axis z . Let α denote the specific torsion angle. After the torsion the generators of the cylinder take the shape of the circular propeller of length

$$(1) \quad ds' = \sqrt{l^2 + r^2 \alpha^2}.$$

The specific elongation, if the expression $r\alpha/l$ is not too big, is

$$(2) \quad \frac{1}{l} \left(\sqrt{l^2 + r^2 \alpha^2} - l \right) \cong \frac{r^2 \alpha^2}{2l^2}.$$

For a generator situated at a distance r of a cylindrical axis we denote by χ the specific coefficient of torsion

$$(3) \quad \frac{r^2 \alpha^2}{2l^2} = \chi r^2 \quad \text{where} \quad \chi = \frac{\alpha^2}{2l^2}.$$

We take the expression χr^2 as a measure of the specific elongation to the z -axis, therefore it will represent the component ε_z of the deformation tensor.

In cylindrical coordinates (r, θ, z) , the deformations which are not zero are:

$$(4) \quad \varepsilon_z = \chi r^2, \quad \varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}.$$

In relation (4), u represents radial displacement. Due to the condition of uniform torsion the tangential displacement u_θ will be constant for any constant radius r and will not appear in the component of the deformation tensor.

The generalized Hook's law equations are [2], [5]:

$$(5) \quad \begin{aligned} \sigma_z &= \lambda \Theta + 2\mu \varepsilon_z, \\ \sigma_r &= \lambda \Theta + 2\mu \varepsilon_r, \\ \sigma_\theta &= \lambda \Theta + 2\mu \varepsilon_\theta, \end{aligned}$$

where λ, μ are the Lamé's constants and $\Theta = \varepsilon_z + \varepsilon_\theta + \varepsilon_r$. Substituting the components of the deformations tensor given by (4) into (5) we obtain

$$(6) \quad \begin{aligned} \sigma_z &= (\lambda + 2\mu) \chi r^2 + \lambda \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right), \\ \sigma_r &= (\lambda + 2\mu) \frac{\partial u}{\partial r} + \lambda \left(\chi r^2 + \frac{u}{r} \right), \\ \sigma_\theta &= (\lambda + 2\mu) \frac{u}{r} + \lambda \left(\chi r^2 + \frac{\partial u}{\partial r} \right). \end{aligned}$$

If we project the forces which work on an element of the bar of length dz on the radial direction we have

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr \right) (r + dr) d\theta dz - 2\sigma_\theta dr dz \frac{d\theta}{2} = 0.$$

Thus we obtain the equilibrium equation [2], [5]

$$(7) \quad \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0.$$

Substituting (6) into (7) we get the linear differential equation to determine the radial displacement

$$(8) \quad \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = -\frac{2\lambda}{\lambda + 2\mu} \chi r.$$

The general solution of Eq. (8) is

$$(9) \quad u(r) = Ar + B \frac{1}{r} - \frac{\lambda}{4(\lambda + 2\mu)} \chi r^3,$$

where

$$u_p = -\frac{\lambda}{4(\lambda + 2\mu)} \chi r^3$$

is a particular solution of Eq. (8). Substituting (9) into (6.3) we arrive to

$$(10) \quad \sigma_r = \frac{\lambda\mu}{2(\lambda + 2\mu)} \chi r^2 + 2(\lambda + \mu)A - 2\mu \frac{1}{r^2} B.$$

The constants A and B are found from the boundary conditions $\sigma_r(a) = \sigma_r(b) = 0$:

$$(11) \quad A = -\frac{\lambda\mu}{4(\lambda + \mu)(\lambda + 2\mu)} \chi (a^2 + b^2), \quad B = -\frac{a^2 b^2 \lambda}{4(\lambda + 2\mu)} \chi.$$

The substitution of (11) into (9) and (10) gives

$$(12) \quad u(r) = -\frac{\lambda\chi}{4(\lambda + 2\mu)} \left[r^3 + \frac{\mu(a^2 + b^2)}{\lambda + \mu} r + a^2 b^2 \frac{1}{r} \right],$$

$$(13) \quad \sigma_r = -\frac{\lambda\mu\chi}{2(\lambda + 2\mu)} \left(a^2 + b^2 - r^2 - \frac{a^2 b^2}{r^2} \right).$$

Since replacing the Lamé's constants with Young's modulus E and Poisson's coefficient ν , the relations (12) and (6) can be written as follows:

$$(14) \quad u(r) = -\frac{\chi}{4} \frac{\nu}{1 - \nu} \left[r^3 + (a^2 + b^2)(1 - 2\nu)r + \frac{a^2 b^2}{r} \right],$$

$$(15) \quad \sigma_r = -\frac{\chi}{4} \frac{\nu E}{1 - \nu} \left(a^2 + b^2 - r^2 - \frac{a^2 b^2}{r^2} \right),$$

$$(16) \quad \sigma_z = \frac{\chi}{4} \frac{E}{1-\nu} \left(r^2 - \frac{a^2 + b^2}{2} \nu^2 \right),$$

$$(17) \quad \sigma_\theta = \frac{\chi}{4} \frac{\nu E}{1-\nu} \left[3r^2 - (a^2 + b^2) - \frac{a^2 b^2}{r^2} \right].$$

In particular, the choice $a = 0$, $0 \leq r \leq b$ yields the result from [1] for a circular cylinder

$$(18) \quad u(r) = -\frac{\chi}{4} \frac{\nu}{1-\nu} [r^3 + b^2(1-2\nu)r];$$

$$\sigma_r = -\frac{\chi}{4} \frac{\nu E}{1-\nu^2} (b^2 - r^2),$$

$$(19) \quad \sigma_z = \frac{\chi}{4} \frac{E}{1-\nu^2} \left(r^2 - \frac{b^2}{2} \nu^2 \right),$$

$$\sigma_\theta = \frac{\chi}{4} \frac{\nu E}{1-\nu^2} (3r^2 - b^2).$$

The constant χ is found from the ends condition

$$(20) \quad M = \iint_D \sigma_\theta r d\sigma = \iint_{D(r,\theta)} \sigma_\theta r^2 dr d\theta,$$

where the domain D is the region of the cross section $0 \leq r \leq b$ for a cylinder or $a \leq r \leq b$, $\theta \in [0, 2\pi]$ for a hollow cylinder.

2. Study of the Stresses

From the relation (15), in the case of a hollow cylinder, we obtain that the radial stress σ_r have a maximum value for $r = \sqrt{ab}$ and does not change its sign in the interval $[a, b]$ as shown in Fig. 1. The maximum value of σ_r is

$$(21) \quad \sigma_r^{max} = \frac{\chi}{4} \frac{\nu E}{1-\nu^2} (b-a)^2 \equiv \chi \Lambda \nu b^2 (1-t^2),$$

where

$$(22) \quad \Lambda = \frac{1}{4} \frac{E}{1-\nu^2}, \quad t = \frac{a}{b}, \quad 0 \leq t \leq 1.$$

Using the relations (15) we conclude that the radial displacement does not change its sign in the interval $[a, b]$. The new deformed radius of the tube becomes

$$(23) \quad \begin{aligned} a' &= a + u(a) = a \left[1 - \frac{\chi \nu}{2} (a^2 + b^2) \right], \\ b' &= b + u(b) = b \left[1 - \frac{\chi \nu}{2} (a^2 + b^2) \right]. \end{aligned}$$

For the fixed ends, the inner radius a , the outer radius b and the thickness of the tube $d = b - a$, grow smaller

$$(24) \quad d' = b' - a' = d \left[1 - \frac{\chi \nu}{2} (a^2 + b^2) \right], \quad d' < d.$$

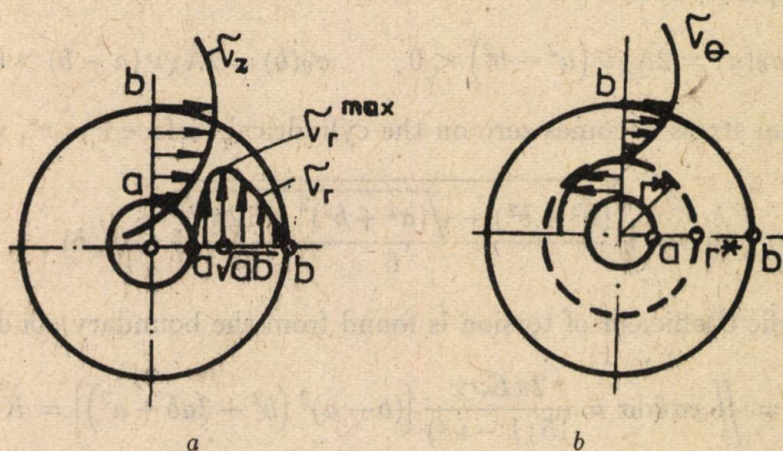


Fig. 1 — Diagram of the stresses $\sigma_r, \sigma_z, \sigma_\theta$.

In the case of circular cylinder the maximum value of the stress σ_r is obtained for $r = 0$ and it is equal to $\sigma_r^{\max} = \chi \Lambda \nu b^2$. This value is bigger than the one in the case of hollow cylinder.

By the expression of the longitudinal stress $\sigma_z(r)$ given by the relation (16) one still observes that it does not change its sign, does not become zero and it increases, so that the extremum value is obtained for $z = a$ and $z = b$, as shown in Fig 1 a. The extremum values of $\sigma_z(r)$ are

$$(25) \quad \sigma_z^{\min} = \frac{\Lambda \chi}{2} [a^2 (2 - \nu^2) - \nu^2 b^2], \quad \sigma_z^{\max} = \frac{\Lambda \chi}{2} [b^2 (2 - \nu^2) - \nu^2 a^2].$$

By the expression of $\sigma_z(r)$ we can calculate the longitudinal forces P , namely

$$(26) \quad P = \iint_D \sigma_z d\sigma = \int_0^{2\pi} d\theta \int_a^b r \sigma_z dr = \frac{\chi E \pi}{8} (b^2 - a^2) = \frac{E \pi \alpha^2 b^4}{16 l^2} (1 - t^2), \quad t = \frac{a}{b}.$$

For the fixed ends, due to the longitudinal deformation the cross section becomes warped. In the case of free ends, the tube grows smaller with $u_z = Pl/(EA) =$

$= E\alpha^2 J_0 / (8\pi)$. Here J_0 is the polar moment of inertia and A is the area of the cross section.

In the particular case of circular cylinder ($a = 0$, $t = 0$) we can see that

$$(27) \quad \sigma_z^{\min} = \chi \Lambda \frac{b^2}{2} \nu^2, \quad \sigma_z^{\max} = \chi \Lambda \frac{b^2}{2} (2 - \nu^2), \quad P = \frac{E\pi\alpha^2 b^4}{16l^2}, \quad u_z = \frac{E\alpha^2 b^2}{16l}.$$

So, we deduce that the longitudinal effort for the tube is smaller than one for a circular cylinder. In the light of this result it is recommended to replace the cylinder by tube to provide a better strength and to spare the material.

From the study of the tangential stress given by the relation (17), we observe that it takes the same absolute value but with opposite signum, as shown in Fig. 1 b for $r = a$, respectively $r = b$,

$$(28) \quad \sigma_\theta(a) = 2\Lambda\chi\nu(a^2 - b^2) < 0, \quad \sigma_\theta(b) = 2\Lambda\chi\nu(a - b) > 0.$$

The tangential stress becomes zero on the cylindrical surface $r = r^*$, where

$$(29) \quad r^* = \sqrt{\frac{(a^2 + b^2) + \sqrt{(a^2 + b^2)^2 + 12a^2b^2}}{6}} \in (a, b).$$

The specific coefficient of torsion is found from the boundary condition

$$(30) \quad M = \iint_D r\sigma_\theta d\sigma = \frac{2\pi E\nu\chi}{15(1 - \nu^2)} [(b - a)^3 (b^2 + 3ab + a^2)] = K\chi D.$$

So, we obtain

$$(31) \quad \chi = \frac{M}{KD}, \quad \text{where} \quad K = \frac{2\pi E\nu}{15(1 - \nu^2)},$$

$$D = b^5 f(t), \quad f(t) = (1 - t^3)(t^2 + 3t + 1), \quad t = \frac{a}{b}.$$

The quantity D will be termed the geometric torsional rigidity. From the study of the $D(t)$ as function of t , $t \in [0, 1]$ we deduce that in the case of the tube the torsional angle is smaller than one in the case of circular cylinder. As we have already mentioned, we recommend to replace the circular cylinder by the hollow cylinder.

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REFERENCES

1. Dragoș L., *Principiile mecanicii mediilor continue*. Ed. Tehnică, București, 1983.

2. Islinski A. Y., *Mechanics of Elastic Solid and Rigid Body*. Nauka, Moscow, 1979.
3. Lupu M., *Mathematical Models of the Mechanics of Continuous Medium*. Press University, Braşov, 1985.
4. Nicoară D., Lupu M., *Study of Deformations of Elastic Hollow Bars for Big Torsion Angle*. Proc. of the 2-th National Conference in Applied Mathematics and Mechanics. Technical University Cluj-Napoca, 1988, pp. 95-98.
5. Timoshenko S. P., Godier J. N., *Theory of Elasticity*. Nauka, Moscow, 1979.

STUDIUL STĂRII DE TENSIUNE ÎN TUBURI ELASTICE LA UNGHIURI DE TORSIUNE MARI

(Rezumat)

Se propune un model matematic pentru studiul torsiunii barelor elastice cilindrice pentru unghiuri de torsiune mari. Se determină tensiunile și se discută comparativ problema torsiunii pentru tuburi și pentru bare cilindrice cu capete fixe sau libere.

1. The first part of the paper is devoted to a review of the literature on the subject of the elasticity of demand for foreign exchange. It is found that the elasticity of demand for foreign exchange is generally found to be inelastic, but that there is considerable variation in the degree of inelasticity among different countries and different periods of time.

THE ELASTICITY OF DEMAND FOR FOREIGN EXCHANGE

The purpose of this paper is to examine the elasticity of demand for foreign exchange in the United Kingdom. It is found that the elasticity of demand for foreign exchange is generally found to be inelastic, but that there is considerable variation in the degree of inelasticity among different countries and different periods of time.

1. INTRODUCTION

The purpose of this paper is to examine the elasticity of demand for foreign exchange in the United Kingdom. It is found that the elasticity of demand for foreign exchange is generally found to be inelastic, but that there is considerable variation in the degree of inelasticity among different countries and different periods of time.

Key words: Elasticity of demand for foreign exchange.

2. THE ELASTICITY OF DEMAND FOR FOREIGN EXCHANGE

There are several factors which may be expected to influence the elasticity of demand for foreign exchange. These factors are: (1) the degree of openness of the economy, (2) the degree of specialization of the economy, (3) the degree of substitution between domestic and foreign goods, and (4) the degree of substitution between domestic and foreign services.

The degree of openness of the economy is a factor which is likely to influence the elasticity of demand for foreign exchange. The more open the economy is, the more likely it is that the demand for foreign exchange will be elastic.

Another factor which is likely to influence the elasticity of demand for foreign exchange is the degree of specialization of the economy. The more specialized the economy is, the more likely it is that the demand for foreign exchange will be inelastic. This is because a specialized economy is more dependent on foreign trade for its supply of raw materials and for its market for its finished products.

The degree of substitution between domestic and foreign goods is another factor which is likely to influence the elasticity of demand for foreign exchange. The more easily domestic goods can be substituted for foreign goods, the more likely it is that the demand for foreign exchange will be elastic.

Finally, the degree of substitution between domestic and foreign services is a factor which is likely to influence the elasticity of demand for foreign exchange. The more easily domestic services can be substituted for foreign services, the more likely it is that the demand for foreign exchange will be elastic.

EFFECTS OF COMPRESSIVE AND TENSILE STRESSES ON THE MAGNETIC BARKHAUSEN NOISE

BY

OCTAVIAN CIOBANU

Abstract. The paper presents an original arrangement of some devices in order to study the effects of compressive and tensile stresses on Barkhausen noise. There are presented some explanations on the nonlinear dependence of noise on stress.

Key words: Barkhausen noise, nondestructive measuring of applied stress.

1. Introduction

There are several magnetic methods for nondestructively measuring applied stress in ferromagnetic materials. Recently, many investigations have been directed toward applying the Barkhausen effect to nondestructive evaluation [1]...[4].

The magnetic Barkhausen effect is dependent on stress as well as the relative direction of the applied magnetic field to the stress direction [1].

Abrupt movements of magnetic domains, as a specimen is magnetized, produce noiselike pulses known as Barkhausen noise in an induction coil placed on the specimen. In a ferromagnetic crystal, domains tend to minimize their overall energy by lying along an easy magnetization direction. The application of an elastic stress to the crystal leads to a modification of the preferred domain orientation due to the magnetostrictive effect.

When subjected to a tensile stress in a region of positive magnetostriction, these domains will re-orient to the $\langle 100 \rangle$ directions that lie to the stress axis [2].

Alternatively, compression encourages a domain alignment which is perpendicular to the stress axis.

2. Experimental Approach

The Barkhausen noise technique is implemented with the arrangement shown schematically in Fig. 1. The magnetic field is applied to a specimen with an electromagnet. The electromagnet 1 is subjected to a sinusoidal current.

The abrupt movements of the magnetic domains are detected with an inductive coil 2 placed on the specimen 3.

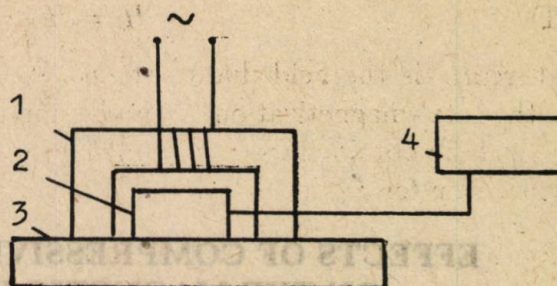


Fig. 1

The induced voltage in the inductive coil 2 was preamplified and fed through a band filter in a selective voltmeter 4.

Certain features of the signal, such as the maximum amplitude, root mean square or the mean amplitude of the Barkhausen noise are used to determine the stress state in the specimen.

The selective voltmeter used (type VS TT 1302) had measured the mean amplitude of the signal.

In this investigation, experimental data were obtained from OLC 35 (Romanian standard) steel specimens. The specimen for compressive stresses was 70 mm long, 15 mm thick and 20 mm wide in the gauge section. The specimen for tensile stresses was 500 mm long, 3 mm thick and 40 mm wide in the gauge section.

To minimize initial residual stresses, the specimens were annealed at 590° C and furnace cooled. Uniaxial stresses were applied to the specimens using mechanical loading fixtures.

3. Results and Discussion

Fig. 2 shows a typical stress dependence of the Barkhausen noise amplitude, when the magnetic field is parallel to the applied stress direction. The Barkhausen noise amplitude thus determined was reproducible as long as the applied magnetic field was unchanged, i.e., the shape, frequency and magnitude of magnetizing current applied to the electromagnet were unchanged.

As shown in Fig. 2, the tensile stress dependence has a greater sensitivity of the signal with respect to compressive stress dependence.

The effects of stresses on the Barkhausen noise can be explained using the Schneider-Cannell-Watts model [4]. The stress affects the magnetization because it acts effectively as if it were a magnetic field acting along the

magnetization axis.

The internal domain effective field H_i is

$$(1) \quad H_i = H + H_\sigma - D_\sigma M,$$

where H_σ is the field due to stress, D_σ is the demagnetization parameter and M is the total magnetization (H is the applied field). The demagnetization parameter D_σ can be obtained from minimizing the domain free energy density. The resulting expression for D_σ is

$$(2) \quad D_\sigma = \frac{3\lambda\sigma}{\mu_0 M_s^2},$$

where λ , σ , μ_0 are magnetostriction, stress and permeability and M_s is the local domain magnetization.

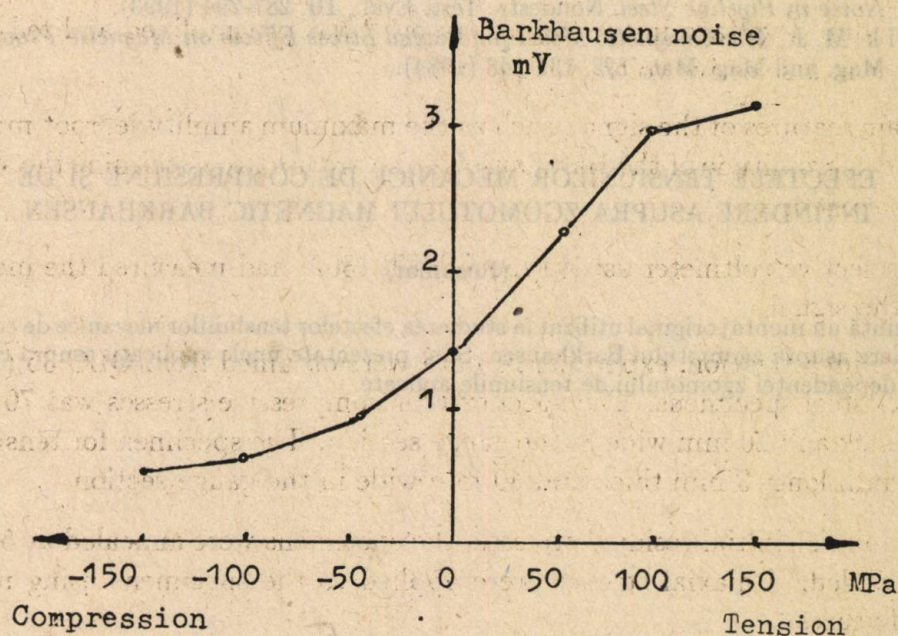


Fig. 2

The demagnetization field $-D_\sigma M$ from (1) physically can be viewed as a demagnetization response of the material owing to the placement of domain walls, their position being responsive to the stress.

For compressive stresses, D_σ is positive and decreases almost linearly as stress goes to zero. For tensile stresses, D_σ increases moderately [4], [3].

This means that in a ferromagnetic material, the internal domain effective field will exhibit large changes when the stress passes from tensile to compressive values.

4. Conclusion

The study reveals an experimental evidence which indicates a direct dependence

of the Barkhausen noise amplitude on stress. The nonlinear dependence on stress can be explained using a theory of the demagnetization response of the material.

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REFERENCES

1. K w u n H., *Investigation of the Dependence of Barkhausen Noise on Stress and the Angle Between the Stress and Magnetization Directions*. J. of Mag. and Mag. Mat., 49, 235-240 (1985)
2. J a g a d i s h C., *The Influence of Stress on Surface Barkhausen Noise Generation in Pipeline Steels*. IEEE Trans. on Mag., 25, 3452-3454 (Sept. 1989).
3. D h a r A., *Effects of Magnetic Flux Density and Tensile Stress on the Magnetic Barkhausen Noise in Pipeline Steel*. Nondestr. Test. Eval., 10, 287-294 (1993).
4. S a b l i k M. J., *Micromagnetic Model for Biaxial Stress Effects on Magnetic Properties*. J. of Mag. and Mag. Mat, 132, 131-148 (1984).

EFECTELE TENSIUNILOR MECANICE DE COMPRESIUNE ȘI DE ÎNTINDERE ASUPRA ZGOMOTULUI MAGNETIC BARKHAUSEN

(Rezumat)

Se prezintă un montaj original utilizat la studierea efectelor tensiunilor mecanice de compresiune și de întindere asupra zgomotului Barkhausen. Sunt prezentate unele explicații asupra caracterului neliniar al dependenței zgomotului de tensiunile aplicate.

THE USE OF CLASSICAL METHODS IN THE
OBTAINING OF SEMICONDUCTOR THIN
FILMS OF THE $\text{Bi}_{2-x}\text{Sb}_x\text{Te}_{3-y}\text{Se}_y$ TYPE

BY

ALEXANDRINA JEFLEA, EMIL LUCA and SOFIA MELINTE

Abstract. The essay contains estimations over evaporation speeds of the components resulted from dissociating $\text{Bi}_{2-x}\text{Sb}_x\text{Te}_{3-y}\text{Se}_y$ type materials, taking into consideration vaporisation coefficients in the event of an infinite charge evaporation. The debate over finite charge evaporation leads to the introducing of the *constituent exhausting time* term. The obtaining of stratified thin films can be explained with regard to the relative exhausting periods. The instant superposed evaporation method eliminates this drawback. The frequency of charge falling into the evaporator and the dimension of the charge.

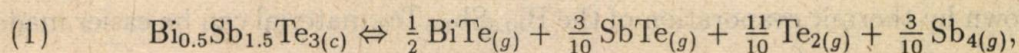
Key words: thin film, thermoelectricity, transport phenomena.

1. Introduction

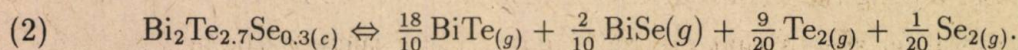
The essay contains estimations over evaporation speeds of the components resulted from dissociating $\text{Bi}_{2-x}\text{Sb}_x\text{Te}_{3-y}\text{Se}_y$ type materials, taking into consideration vaporization coefficients in the event of an infinite charge evaporation. The debate over finite charge evaporation leads to the introducing of the *constituent exhausting time* term. The obtaining of stratified thin films can be explained with regard to the relative exhausting periods. The instant superposed evaporation method eliminates this drawback. The frequency of charge falling into the evaporator is given by the temperature of the evaporator and the dimensions of the charge.

2. Analysis and Discussion

Materials on Bi_2Te_3 , under the form of $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$ and $\text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3}$ basis present great difficulties during the thin film achieving process. Taking into consideration the dissociation type for Bi_2Te_3 , Bi_2 , Se_3 and Sb_2Te_3 exposed in [1], we propose a dissociation equation for $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$ as it follows:



and for $\text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3(c)}$ we have



Now we are able to calculate the steam pressures of the resulted components, using the thermodynamic coefficients from [2]

$$\lg p_{\text{torr}} = -\frac{A_i}{T} + B_i,$$

resulted from the following:

$$(3) \quad \ln p^* = -\frac{\Delta_e H_{(298)}^0}{\nu RT} + \frac{\Delta_e S_{(298)}}{\nu R} + \frac{1}{\nu RT} \int_{298}^T \frac{\Delta C_p}{T} (dT)^2.$$

Knowing that evaporation is efficient when steam pressure is higher than 10^{-2} torr, we may find the evaporation temperature of each constituent:

i) for $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$

$$(4) \quad \begin{array}{ll} t_v \text{ for BiTe} = 585^\circ\text{C}, & t_v \text{ for Sb}_4 = 563^\circ\text{C}, \\ t_v \text{ for SbTe} = 569^\circ\text{C}, & t_v \text{ for Te}_2 = 545^\circ\text{C}; \end{array}$$

ii) for $\text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3}$

$$(5) \quad \begin{array}{ll} t_v \text{ for BiTe} = 585^\circ\text{C}, & t_v \text{ for Te}_2 = 575^\circ\text{C}, \\ t_v \text{ for BiSe} = 600.4^\circ\text{C}, & t_v \text{ for Se}_2 = 646.7^\circ\text{C}. \end{array}$$

After analyzing the partial pressures of the components, we may calculate the minimum temperature of the evaporator for which the evaporation of any constituent is efficient, and this is 646°C for $\text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3}$ and 585°C for $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$. We can pass to the calculus of the evaporation speed, having the steam saturation pressure expressed in a formula. According to the Hertz - Knudsen law [3], the evaporation speed is given by

$$(6) \quad \frac{dN_e}{A_e dt} = (2\pi mkT)^{-\frac{1}{2}} (p^* - p).$$

This theoretical evaporation speed could not be obtained, even after special cleaning of the evaporation surface and within vacuum. Knudsen introduced the evaporation coefficient α_v . We deduce, from experimental observations, that α_v is not the same for all the components that resulted from the dissociation. The relative values of the α_v coefficient are estimated in [4]. Thus estimated, the evaporation coefficients correspond qualitatively to the experimental observation, meaning that setting down by thermic evaporation of the $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$ material can be easier made

to correspond with the basic material, that is the evaporation coefficients have values that are closer among them, in comparison with those of the $\text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3}$ material.

The evaporation speeds that are recalculated using these α coefficients lead to the obtaining of some non-stoechiometric steam fascicles.

The *exhausting period* of the charge in different constituents appears, in the event of a finite charge of material, as a result of the different evaporation speeds of the components. Thus, the exhausting period for a component with a x weight, for one mol of evaporated material with the evaporation surface of 1 cm^2 , will be

$$(7) \quad \tau_x = \frac{N_A x}{N_{\text{ex}}},$$

in which N_A is Avogadro's number; x is the weight of the compound in the dissociated material; N_{ex} is the evaporation speed of the respective component

$$(8) \quad N_{\text{ex}} = (2\pi mkT)^{-\frac{1}{2}} \alpha_x p_{\text{ex}}^*,$$

in which p_{ex}^* is the steam partial pressure of the component

$$(9) \quad p_{\text{ex}}^* = x 10^{-\frac{A_i}{T} + B_i};$$

A_i and B_i are evaporation constants.

There is very important to determine the relative exhausting periods for different components. The exhausting periods within the $\text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3}$ material, in relation with the one for selenium, will be:

$$\tau_{\text{Te}_2} = 1.6 \tau_{\text{Se}_2},$$

$$(10) \quad \tau_{\text{BiTe}} = 2.5 \tau_{\text{Se}_2},$$

$$\tau_{\text{BiSe}} = 2.14 \tau_{\text{Se}_2}.$$

We relate the exhausting periods within the $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$ material to SbTe :

$$\tau_{\text{Sb}_4} = 1.296 \tau_{\text{SbTe}},$$

$$(11) \quad \tau_{\text{Te}_2} = 1.06 \tau_{\text{SbTe}},$$

$$\tau_{\text{BiTe}} = 1.53 \tau_{\text{SbTe}}.$$

Calculating these exhausting periods for these constituents, for a charge of 1, with the evaporating surface of 1 cm^2 enough. One can see that exhausting periods for $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$ are closer than those for $\text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3}$. This result is confirmed in [6], where the authors, by means of spectrometric analysis, emphasize the change in composition of the fascicle evaporated from the Bi_2X_3 (X can be Se or Te) material during the evaporation.

The differences that show up between evaporation periods of the less volatile components and the evaporation time calculated for the material in a non-dissociated state could be the consequence of two causes:

- i) the value of the evaporation coefficient for the non-dissociated material is less than one unit;
- ii) the evaporation coefficient of the less volatile component is increased to the value calculated for the initial composition as the volatile components evaporate; thus it may not be considered constant till the end of the charge evaporation. Because in the following paragraphs we are interested only in relative values for exhausting periods, we can accept this disagreement.

Within the flash (or instant) evaporation method the arrival of the material into the evaporator is discreet, in small quantities, and the charge is completely evaporated until the next one arrives. When using the $\text{Bi}_{2-x}\text{Sb}_x\text{Te}_{3-y}\text{Se}_y$ type material, because of their dissociating in several components which have different evaporation speeds, and because the steam fascicle is not stoecheometric, the exhausting periods for the components are different, as we have shown, and in the last stage of the evaporation of a finite charge of material this charge will have a deficit in more volatile components, and the deposited film will obviously have another composition. That is why we propose two methods of instant evaporation:

- 1) an instant evaporation method in selenium atmosphere;
- 2) an instant superposed evaporation method.

The instant evaporation method has generally given good results for the mentioned materials, in conditions of evaporating and condensing a charge in order to achieve a mono-layer film. The dimensions of the charge that arrived into the evaporator, for which the deposited film should be close to the value c of the constant of the crystalline net, that is a thickness of about 30 Å, in the system consisting of evaporator-punctual source-plane substrate, positioned at distance h from the source, [8] will be

$$(12) \quad t = \frac{V h}{2\pi(h^2 + \delta^2)^{\frac{3}{2}}},$$

where t is the thickness of the film at distance δ from the center of the deposit, h is distance between evaporator and, V is the volume of evaporated material. Supposing the evaporated particle is a sphere, the dependence between the thickness of the film and the charge radius will be

$$(13) \quad r = \left(1 + \frac{\delta^2}{h^2}\right)^{\frac{1}{2}} \left(\frac{3th^2}{2}\right)^{\frac{1}{3}}.$$

Very fine dusts are needed for the mono-layer evaporation, r having values between 0.25...0.33 mm, and that requires a fine sorting of the dust and a continuous high-frequency vibration of the feeding device of the evaporator.

The method derived from that of instant evaporation is the superposed or continuous instant evaporation. Starting from the observations published in [9] we have adopted the flash method for these thermoelectric materials.

From calculating exhausting times in components it results that only in the first 2/3, respectively 2/5 of the total period of charge evaporation the steam fascicle will contain all the components. It is from here that we deduced the frequency of bringing the charges into the evaporator, so that the evaporation of the last part of charge $n - 1$ will superpose with the first part of charge n , etc. This way there can be provided a minimum fluctuation in the concentration of the evaporated fascicle of material. The falling frequency ν_c is the inverse of the falling interval θ_c , which we consider to be equal to the exhausting time of the most volatile element ($\frac{2}{3} \tau_{\max}$, respectively $\frac{3}{5} \tau_{\max}$), and that is:

$$(14) \quad \begin{aligned} \theta_1 &= \frac{2}{3} \tau_{\text{BiTe}}, & \text{for } \text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3}, \\ \theta_2 &= \frac{2}{3} \tau_{\text{BiTe}}, & \text{for } \text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3. \end{aligned}$$

First

$$(15) \quad \theta_1 = \frac{10}{9} r \tau_{\text{Se}_2}, \quad \theta_2 = \frac{3.06}{5} r \tau_{\text{SbTe}}.$$

Noticing that τ_{Se_2} and τ_{SbTe} is dependent of the temperature of the evaporator, we have

$$(16) \quad \frac{10}{9} r \tau_{\text{Se}_2}(T_{\min}) < \tau_{c1} < \frac{10}{9} r \tau_{\text{Se}_2}(T_{\max})$$

and

$$(17) \quad \frac{3.06}{5} r \tau_{\text{SbTe}}(T_{\min}) < \tau_{c2} < \frac{3.06}{5} r \tau_{\text{SbTe}}(T_{\max}).$$

These formulas restrict the boundaries of the falling interval for r radius spherical charges.

Considering reasonable values of around one second for the falling interval, one can express the particles dimensions [10] as

$$(18) \quad r_1 = \frac{9}{10} \frac{\tau_{c1}}{\tau_{\text{Se}_2}(T)} \quad \text{and} \quad r_2 = \frac{5}{3.06} \frac{\tau_{c2}}{\tau_{\text{SbTe}}(T)}.$$

Thus we are able to restrict particle dimensions within the next ranges:

$$(19) \quad \begin{aligned} \frac{9}{10} \frac{1}{\tau_{\text{Se}_2}(T_{\max})} < r_1 < \frac{9}{10} \frac{1}{\tau_{\text{Se}_2}(T_{\min})}, \\ \frac{5}{3.06} \frac{1}{\tau_{\text{SbTe}}(T_{\max})} < r_2 < \frac{5}{3.06} \frac{1}{\tau_{\text{SbTe}}(T_{\min})}. \end{aligned}$$

3. Conclusion

The method can be easily applied to an instant evaporation device, which has the possibility of adjusting the frequency of the charge falling into the evaporator. The dust must be shifted at the dimension established by the temperature of the evaporator. The films obtained through this method present properties superior to those obtained by simple or by flash evaporation. X-ray diffraction analysis confirm the superiority of the method.

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REFERENCES

1. Goltzmann B. M., Kudinov V. A., Smirnov I. A., *Poluprovodnikovie termoelectriceskie materiali na osnove Bi_2Te_3* . Nauka, Moskva, 1972.
2. * * *Fiziko-himiceskie svoistvo poluprovodnikovih vescestv. Spravocinik*. Nauka, Moskva, 1979.
3. Kundsen M., *Ann. Physik*, **52**, 105 (1917).
4. Jeflea A., Luca E., Antonescu I., *An. Șt. Univ. „Al. I. Cuza“, Iași* (1990).
5. Jeflea A., Teză de doctorat, Univ. „Al. I. Cuza“, Iași, 1990.
6. Lidorenko N. S. (coord.), *Plenocinie termoelementi. Fizika i primenenie*. Nauka, Moskva, 1985.
7. Jeflea A., *An. Șt. Univ. „Al. I. Cuza“, Iași* (1990).
8. Günther K. G. Z., *Naturforsch.*, **16 a**, 279 (1961).
9. Ellis S. C., *J. Appl. Phys.*, **38**, 2906 (1967).
10. Jeflea A., Contract de cercetare științifică, Inst. Polit Iași, nr. 4447/1981.

UTILIZAREA DE METODE CLASICE PENTRU OBTINEREA STRATURILOR SUBȚIRI SEMICONDUCTOARE DE TIPUL $Bi_{2-x}Sb_xTe_{3-y}Se_y$

(Rezumat)

Lucrarea conține aprecieri asupra vitezelor de evaporare a compușilor rezultați din disocierea materialelor de tipul $Bi_{2-x}Sb_xTe_{3-y}Se_y$ cu luarea în considerare a coeficienților de vaporizare, în cazul unei evaporări din șarjă infinită. Discuția evaporării din șarjă finită duce la introducerea termenului de *timp de epuizare în componenți*. În funcție de timpii relativi de epuizare în componenți se explică obținerea straturilor subțiri de tip stratificat. Eliminarea acestui neajuns se face folosind metoda evaporării instantanee-suprapusă. Frecvența de cădere a șarjei în evaporator este dată de temperatura evaporatorului și de dimensiunea șarjei.

THE STRUCTURE AND PHOTOCONDUCTIVE PROCESS IN CdS/Cu₂S SOLAR CELLS

BY

SOFIA MELINTE and ALEXANDRINA JEFLEA

Abstract. This essay presents the technology for producing CdS/Cu₂S solar cells with heterojunction. The thin films were deposited on metallic substrate. Different ways to increase the efficiency of the conversion are shown.

Key words: thin films, fotovoltaic effect, direct conversion.

1. Introduction

During the last years the thin films solar cells began to compete more and more with the Si cells. The most studied [1]...[7] and the most promising (of them) are the heterojunction CdS/Cu₂S solar cells with polycrystalline films.

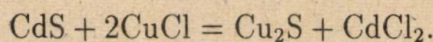
The greatest advantage of CdS cells consists in the relatively low cost of both the process of fabrication and the materials. Their shortcomings consist in their relatively low efficiency and their instability. The researches meant to increase the conversion efficiency of the solar energy in to electric energy and to enlarge their stability in time by means of different technologies and thermal treatments.

In this work the experimental results are presented concerning the realization of CdS/Cu₂S solar cells.

2. Experimental Results

The structure CdS/Cu₂S solar cells is presented in Fig. 1. On a glass or metallic substrate, another metallic layer is deposited, above on a CdS layer deposited by vacuum evaporation. The thickness of CdS layer varies between 2 and 30 μm . On this structure a Cu₂S layer was chemically deposited by using a CuCl – NH₄Cl solution. In order to obtain reproducible results it is necessary that the two substances entering the solution should be in thermodynamic equilibrium [7]. The CuCl solution used for the preparation of Cu₂S layer was hold at 90°. The reaction of formation of cuprous

sulfide is



Above the Cu_2S layer the second electrode was deposited under the form of a grid. The powder CdS is evaporated at about 1000°C on a heated substrate. The temperature of the substrate changed between 150°C and 300°C , depending on its nature.

The thickness of layer was measured by two methods: gravimetric and interferometric (with the Linnik interferometer). Similar values were obtained by these two methods. The resistance of the CdS layer changed between 0.12 and $7 \Omega/\square$ and that of Cu_2S layer was within the range $1.5 \cdot 10^{-2} \dots 5.2 \cdot 10^{-1} \Omega/\square$ depending on the thickness of the layer.

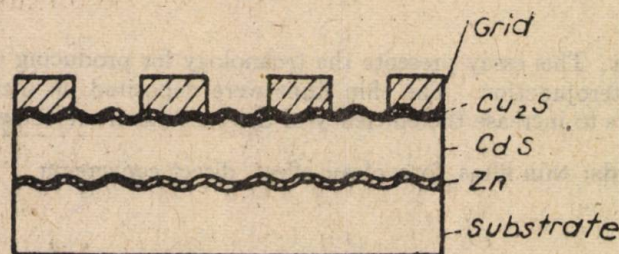


Fig. 1 — The structure of CdS/ Cu_2S solar cells.

The second electrode, made of gold, was deposited above the structure of the cell, at room temperature, by thermal evaporation in vacuum.

The thermal treatment was performed first in vacuum then in air. During the dipping process and the thermal treatment, the copper diffused into the CdS layer which results in the decrease of the resistivity and the increase of the photosensitivity. The thermal treatment was meant to increase the stability of the heterojunction, as well as to produce the diffusion of Zn and Cu ions into the CdS layer which leads to the decrease of the resistance of Cu_2S layer and to the release of the water from the heterojunction.

By microscope studies of the superficial structure of the CdS layer, a non-uniform granular structure and a substance blasting were noticed. These defects are negatively affecting the performances of the cell.

The solar cells thus prepared can produce the conversion of the solar energy into electric energy. The mechanism of the appearance of both photocurrent and photovoltage in such a structure is schematically represented in Fig. 2:

1. The photons are reflected by the grid.
2. Some of the photons cross the structure of the cell if it is transparent.

3. The photons that penetrate into the cell and have enough energy create an electron-hole pair.

4. The electrons having the energy higher than the gap energy also create pairs of carriers and the energy in excess is dissipated within the cell as heat. The electron-hole pair created by the photons diffuse within the heterojunctions where they are separated by the internal electric field. The electrons are directed toward the n region and the holes toward the p region, so that the junction gets polarized.

From this figure we can see that not all the photons falling on this structure generate photocurrents: some of them are reflected, some cross the cell and only part of them produces free carriers. Not all these carriers are directed by the internal electric field. Among the carriers generated within the p region only those generated at a distance from the space charge region S_S less or equal to the diffusion length of the electrons L_n will suffer the action of the internal electric field E_i . All the charge carriers generated within the space charge region will be oriented by the internal field E_i .

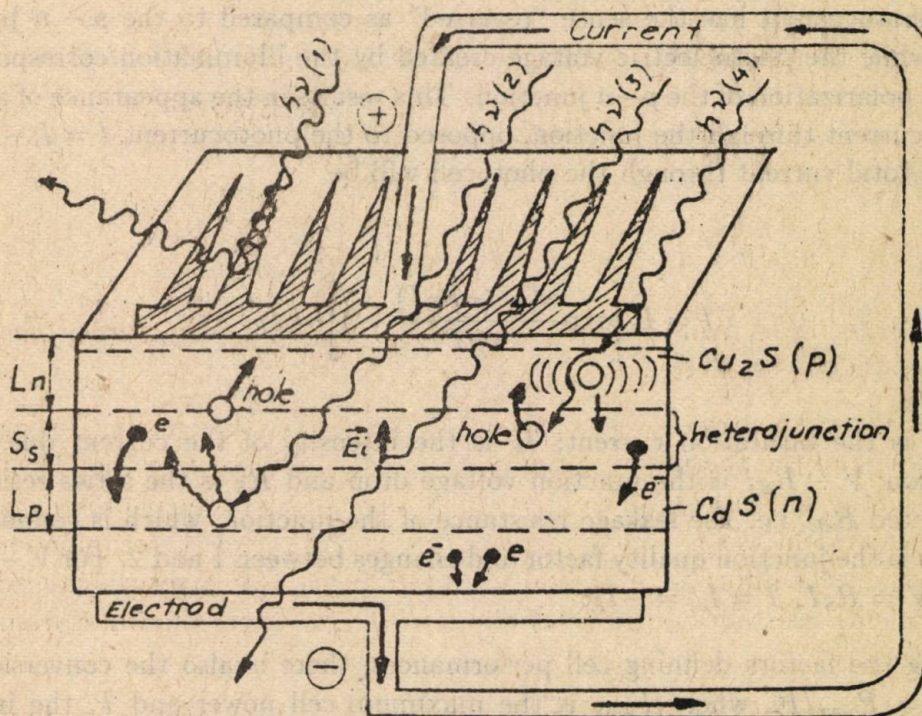


Fig. 2 — The illuminated CdS/Cu₂S cell.

Among the carriers generated within the (CdS)- n region, only those situated within a distance from the space charge region S_S less or equal to the diffusion length of the holes will suffer the influence of the field E_i .

In the case of CdS/Cu₂S cells, the diffusion length in Cu₂S varies from 0.2 to 0.3

μm [7]. Therefore, if the distance from the generation point to the space charge region is smaller than the diffusion length (the distance within which the concentration of the photocarriers decreases e times due to the recombination $\Delta n \sim e^{x/l}$ then the carriers can reach the junction by diffusions.

Therefore the internal electric field E_i (which has the sense from n^- to p^+) facilitates the passage through the space charge region of the disequilibrium minority carriers generated as a result of the photogeneration within p and n regions. This transport of the electric charges leads to a positive charging of the p region and a negative one for the n region. This results in the generation of the photocurrent and of the electric voltage across the cell.

In the case of the open circuit or when the load resistance R_{ex} is much greater than the internal resistance R_i the current through the external circuit is almost zero and the terminal voltage is maximum and equal to V_{OC} . In the case of a short-circuit, for $R_{\text{ex}} \ll R_i$, the terminal voltage is zero and the current is maximum and equal to I_{sc} .

The photocurrent has the sense "reversed" as compared to the $p-n$ junction current, while the photoelectric voltage created by the illumination corresponds to the direct polarization of the $p-n$ junction. This results in the appearance of a direct injection current through the junction, opposed to the photocurrent $I = I_i - I_p$ and hence the total current through the photocell will be

$$I = I_s \left[\exp \frac{e(V - R_S I)}{\alpha k T} - 1 \right] - I_F,$$

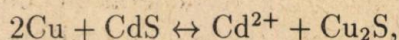
where I_s is the saturation current; I_F is the intensity of the current due to the illumination; $V - R_S I$ is the junction voltage drop and R_S is the series resistance. We neglected R_{sh} , i.e. the leakage resistance of the junction, which is connected in parallel, a is the junction quality factor and changes between 1 and 2. For $V - R_S I = 0$, i.e. $V = R_S I$, $I = I_{sc} = -I_F$.

Among the factors defining cell performances, there is also the conversion efficiency $\eta = P_{\text{max}}/P_i$, where P_{max} is the maximum cell power and P_i the incident radiation power.

Aiming at obtaining a high efficiency, an important part is played by the formation (or generation) of CdS-Cu₂S heterojunction. The Cu₂S layer was obtained on CuCl ionic solution what is very unsteady and gets oxidized in the presence of oxygen solved into water.

If a plate with a CdS thin layer is introduced into a CuCl solution, Cu⁺ goes by oxidation to Cu²⁺ and on the contact surface the following equilibrium are estab-

lished:



In the case of a great concentration of Cu^{2+} a rapid and a non uniform deposition of CuS coating takes place. For the less concentrated solution, the Cu^+ ions slowly oxidized and a uniform CuS coating is formed. It is well known that, simultaneously with the formation of calcite (Cu_2S) other compounds are formed, like: covellite (CuS), djurleita ($\text{Cu}_{1.96}\text{S}$) and digenite ($\text{Cu}_{1.8}\text{S}$) [8], [9]. The above mentioned compounds are formed depending on the concentration of $\text{CuCl} - \text{NH}_4\text{Cl}$ and they give rise to Schottky defects within the crystalline lattice of the cuprous sulfide and play an important part in its electric properties especially affecting the short-circuit current. By trials (tests) the most efficient deposition conditions of the Cu_2S layer were chosen.

The results are the subjects of some other works.

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REFERENCES

1. Bordure G., Henry M. O., Jacquemin J. L., Savelli M., Photovoltaic Energy Conference, Berlin, 1979.
2. Rotwarf A., Barnett A. M., IEEE Trans. on Electron Devices, Ed-24, 4, 381 (1977).
3. Melinte S., Jeflea A., Mateescu N., *The Diffusion Length of Minority Carriers in CdS Films Used for Solar Cells*. J. of the Less-Common Metals, 95, 99-103 (1983).
4. Melinte S., Jeflea A., *The Thin Film MIS Solar Cells Based on CdS*. Proceedings of 4-th Miami International Conference on Alternative Energy Sources (Ed by T. N. Veziroglu), Vol. 3, "Solar Power Applications Alcohols", 1982, pp. 117-127.
5. Melinte S., Jeflea A., Moise M., Mateescu N., *The Study of the Instability of CdS-Cu₂S Solar Cells*. Proceedings of the International Symposium-Workshop on Renewable Energy Sources (Part A, Ed. by T. N. Veziroglu), Elsevier Science Publ., Amsterdam, 1984, pp. 269-275.
6. Melinte S., Jeflea A., *CdS-Cu₂S-Tin Film Solar Cells*. Bul. Inst. Polit. Iași, XXVII (XXXI), 3-4, s. I, 119-123 (1981).
7. Aerschodt A. E. V., Reinhartz K. K., Proc. of ECOSEC, 95 (1970).
8. David I. P., Marinuzzi S., Proc. of ECOSEC, 81 (1980).
9. Tevelde T. S., Solid State Electrons, 1, 1305 (1973).

STRUCTURA ȘI PROCESELE FOTOCONDUCTIVE ÎN CELULELE SOLARE PE BAZĂ DE CdS-Cu₂S

(Rezumat)

Se prezintă tehnologia de obținere a celulelor solare cu heterojoncțiune pe bază de CdS-Cu₂S. Straturile subțiri sunt depuse pe substrat metalic și se prezintă diferitele moduri de creștere a eficienței conversiei.

ON SOME ELECTRO-OPTICAL INTERFACE PROPERTIES AT MIXED SEMICONDUCTORS HETEROSTRUCTURES WITH PHOTOSYNTHETIC PIGMENT

II. OPTICAL INTERFACE ABSORPTION PHENOMENA IN MULTILAYER-TYPE HETEROJUNCTIONS

BY

MAGDA GHERGHEL

Abstract. This paper presents the experimental results concerning the study of the photoelectric behaviour of a multijunction type multilayer mixed heterostructure (solar cell system tandem) having the form: $\text{SnO}_x/\text{CdS: Na-Cu}_x\text{S: Cu-Al-Chl/Cu}$.

The experimental results show that both the photosynthetic pigments (chlorophyll) and the I_a and I_b group impurities (Na, Cu, Li, K) are able to improve the photoeffects at $\text{CdS-Cu}_x\text{S}$ junction, by enhancing its photovoltages and photocurrents/cm², $U_{CD} = 350\ldots 575$ mV and $J_{SC} = 10.5\ldots 15$ mA/cm². Also have been studied the methods in which the temperature, the concentration of the dopants solutions and the external electric fields intervene in the interface physical processes of this mixed heterostructure, as well as the optical absorption mechanism and the phonons scattering mechanism at this type of heterostructure.

Key words: tandem cell system, photovoltaic properties, Schottky barrier, quantal tunneling effect, optical interface absorption mechanism.

1. Introduction

One of the developing directions of the research program on non-conventional energy sources is that including the improvement of the non-conventional energy sources performances based on the photovoltaic effect. In this context, a special attention is given to the technologies for accomplishment of the heterojunctions and heterostructures with a Schottky barrier made up of thin photoconductive layers. This paper presents the experimental results concerning the study of the photoelectric behaviour of the multijunction-type multilayer mixed heterostructure (solar cell system tandem) having the form: $\text{SnO}_x/\text{CdS: Na-Cu}_x\text{S: Cu-Al-Chl/Cu}$.

At the same time, we have studied the optical absorption mechanism as well as the photons scattering mechanism at this type of heterostructure, the knowledge of this phenomenon being extremely important for the understanding of the physical processes at the metal-semiconductor interface with a view to improve the photovoltaic qualities at these heterostructures.

2. Experimental Results and Discussions

On some previous papers [1]–[3], we have underlined the fact that to improve the photovoltaic performances of CdS-Cu_xS-type heterostructures accomplished by the method of the liquid phase reaction, we have directed our researches on the use of some elements from the I_a and I_b groups (Na, Cu) as dopants in the CdS and Cu_xS layers, respectively, on taking into account that impurities are cheaper and easier to find. On taking into account, as well, the considerable photoconducting properties noticed at the photosynthetic pigments, we have tried to insert between the doped layers from the photojunction, some thin layers of microcrystalline chlorophyll (Chl) obtained by electrodeposition, sputtering and dipping: finally, there resulted a mixed multijunction of SnO_x/CdS: Na-Cu_xS: Cu-Al-Chl/Cu type [1]–[3]. Thin microcrystalline films have been prepared by sputtering, electrodeposition and dipping methods described in details in [1]–[3], having a thickness of 10...20 μm and a mean resistivity of 3.10¹¹ Ωcm. For this mixed photoelement, we have noticed an increase of both photocurrents and photovoltages, i.e. $U_{CD}=350...450$ mV and $J_{SC}=10...10.5$ mA/cm² [1]–[2].

The photoelectric measurements to plot the photoelectric characteristic $J_L = f(h\nu)$ (J_L representing the illumination current or the photocurrent), has led, as well, to the determination of the Schottky barrier height at the Al/Chl interface. According to Crowell [4], the quantum efficiency, thus the photocurrent at a structure M-S-M, depends on the photons energy, according to the relation

$$(1) \quad \eta \sim (h\nu - h\nu_0)^2,$$

for $h(\nu - \nu_0) > 3kT$. Since the energy $h\nu_0$ corresponds to the Schottky barrier height,

$$(2) \quad h\nu_0 = eV_B = \phi_B.$$

The photocurrent resulted after the bilateral illumination of the Al-Chl-Cu heterostructure is also proportional with the square of the energy of the incident photons

$$(3) \quad J_L \sim (h\nu - eV_B)^2.$$

From (2) and (3) there results the dependence $\sqrt{\eta} \sim h(\nu - \nu_0)$ and the dependence of the photocurrent J_L , respectively, function of the energy of the incident radiation, $\sqrt{J_L} \sim (h\nu - \phi_B)$.

On representing graphically the value of the square root from the mean values J_L , experimentally obtained for the illumination current (photocurrent), function of the

energy of the photon flow $h\nu$, with which the bilateral illuminating of the Al-Chl-Cu heterostructure has been accomplished, we have obtained a straight line for values of the energy higher than $h\nu_0$ (Fig. 1). By extrapolating this straight line up to the intersection with the energy axis $h\nu$, mean values for the barrier height ϕ_B ranging in the interval 0.95...1.15 eV have been obtained in the intersection point (Fig. 1).

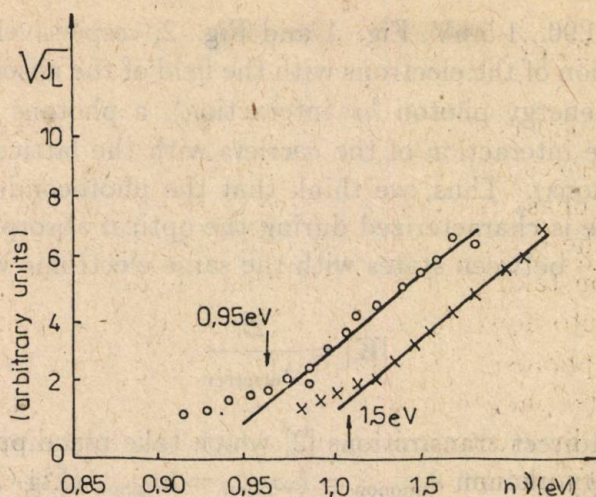


Fig. 1 — The measured photoelectric characteristic.

The study of the spectral characteristics, $U_{CD} = f(\lambda)$, for these types of devices, made evident that the action spectrum of the photoconduction closely follows all the times, the absorption spectrum of microcrystalline chlorophylls, a shift of the red maximum toward the higher wavelength being specific (see Fig. 2).

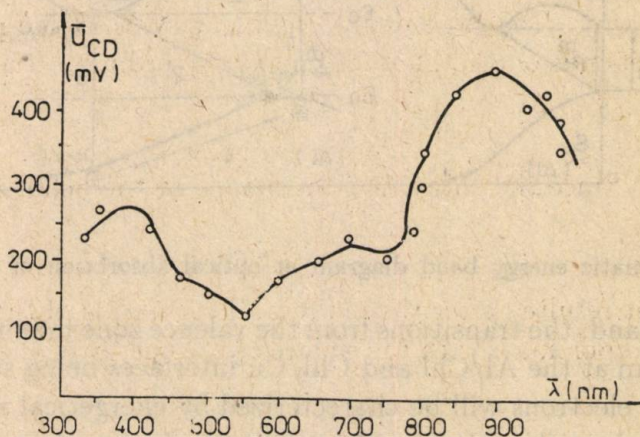


Fig. 2 — The $U_{CD} = f(\lambda)$ characteristics.

On taking into account the ϕ_B height of the Schottky barrier at the Al/Chl interface, we consider the existence of a quantum tunnelling effect of the carriers

through the interface states as well as the possibility of returning to the aluminium (Al) layer of the electrons which have climbed over the barrier as a result of their interaction with the phonons of the interface structural lattice. Thus, the aspect of the dependency

$$\sqrt{J_L} = f(h\nu) \quad \text{and} \quad U_{CD} = f(\lambda),$$

in the region $h\nu = 0.90 \dots 1.5$ eV, Fig. 1 and Fig. 2, respectively, make us consider, besides the interaction of the electrons with the field of the absorbed electromagnetic radiation (electron-energy photon $h\nu$ interaction), a photons scattering device on phonons, due to the interaction of the carriers with the lattice oscillations (optical absorption on phonons). Thus, we think that the photoconduction mechanism at the Al/Chl interface is characterized during the optical absorption process both by direct transmissions - between states with the same electronic wave vector

$$|\mathbf{K}| = \frac{2\pi}{\lambda_{\text{electron}}},$$

$\Delta K = 0$, and by indirect transmissions [2] which take place with the absorption or emission of an energy phonon $\varepsilon_{\text{phonon}} = \hbar\omega_{\text{phonon}} = h\nu_{\text{phonon}}$ (Fig. 3). Thus, we assume that the direct transitions, characterized by $\Delta K = 0$ from the valence-band band, to the interface conduction band (Fig. 3), determine the sudden increase of the absorption in the regions for which $0.95 \text{ eV} \leq h\nu \leq 1.82 \text{ eV}$ and $2.70 \text{ eV} \leq h\nu \leq 3.02 \text{ eV}$, (Figs. 1, 2 and Fig. 4).

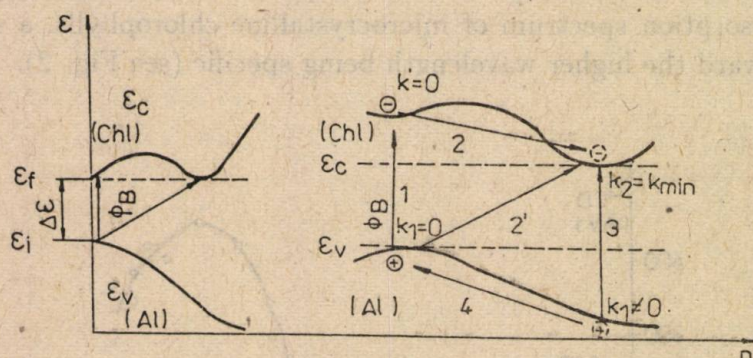


Fig. 3 — Schematic energy band diagram at optical absorption of interface Al/Chl.

On the other hand, the transitions from the valence zone maximum to the conduction zone minimum at the Al/Chl and Chl/Cu interfaces being supposed as indirect ones (Fig. 3), the electrons will be characterized by energetical stages with $K' \neq K$ and interact not only with energy incident radiation $h\nu$ (electron-photon interaction) but also with the interface structural lattice, thus determining the absorption or the emission of a pulse phonon $\hbar\eta$,

$$\mathbf{p} = \hbar\boldsymbol{\eta}, \quad |\boldsymbol{\eta}| = 2\pi/\lambda_{\text{phonon}}.$$

We consider that these two processes in which these three particles, i.e., photon-

electron- phonon, intervene, are characterized by a law of energy conservation [2]:

$$(4) \quad \hbar\omega_e = \varepsilon_f - \varepsilon_i + \varepsilon_{\text{phon}},$$

$$\hbar\omega_a = \varepsilon_f - \varepsilon_i - \varepsilon_{\text{phon}},$$

where $\hbar\omega_e$ and $\hbar\omega_a$ represent the energy of the photon absorbed with the emission of the phonon, and the energy of the photon absorbed with the absorption of the phonon, respectively.

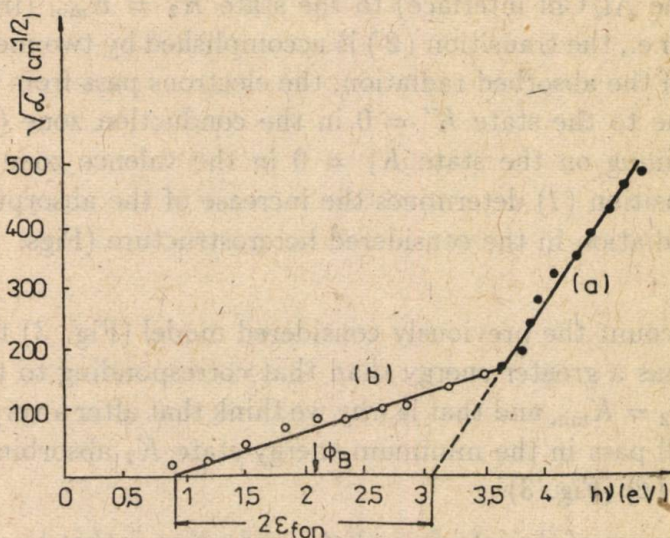


Fig. 4 — The $\alpha = f(h\nu)$ characteristics.

In the relations (4), $\varepsilon_{\text{phon}}$ represents the phononic energy while ε_i and ε_f are the initial and the final energy of the electron, respectively. Consequently, we think that the mechanisms of the optical absorption and of photoconduction at the metal/semiconductor interface from the considered heterostructure are also characterized by the indirect band-band transition in which, in the interaction process between a photon $h\nu = \hbar\omega$ and an electron, an energy phonon $\varepsilon_{\text{phon}}$ intervenes — see relations (4).

At the same time, we consider that in the case of the indirect transitions, the law of pulse conservation at the electron-phonon interaction has the form

$$(5) \quad \hbar\mathbf{K}' = \hbar\mathbf{K} \pm \hbar\boldsymbol{\eta},$$

where $\hbar\mathbf{K}$ represents the electron pulse $|\mathbf{K}| = 2\pi/\lambda_{\text{electron}}$, and $\hbar\boldsymbol{\eta}$ represents the phonon pulse $|\boldsymbol{\eta}| = 2\pi/\lambda_{\text{phon}}$, $\boldsymbol{\eta}$ being the phononic wave vector.

In the following, we assume that the phonon-photon intersection [2] is characterized by the conservation of the pulse and energy:

$$(6) \quad \hbar \mathbf{q} = \hbar \boldsymbol{\eta},$$

$$h\nu = \hbar\omega = \hbar\omega_{\text{phon}},$$

where $\hbar\omega_{\text{phon}} = \varepsilon_{\text{phon}}$. In the relations (6), $\hbar \mathbf{q}$ represents the photon pulse, $\mathbf{q} = 2\pi/\lambda_{\text{photon}}$, and $\hbar\omega$ and $\hbar\omega_{\text{phon}} = \varepsilon_{\text{phon}}$ are the photon energy and the energy of the phonon with which the photon interacts, respectively.

In Fig. 3, we consider the indirect transitions as taking place between virtual states with very short average life, $\tau \approx 10^{-13}$ s, [2], [4], so that we could suppose that the passage of the electrons from the state $K_1 = 0$ (maximum value of the valence zone at the Al/Chl interface) to the state $K_2 = K_{\min}$ (from the interface conduction zone), i.e., the transition (2) is accomplished by two mechanisms. Thus, under the action of the absorbed radiation, the electrons pass from the state $K_1 = 0$ in the valence zone to the state $K' = 0$ in the conduction zone (direct transition (1)), a hole remaining on the state $K_1 = 0$ in the valence zone (Fig. 3). This type of direct transition (1) determines the increase of the absorption coefficient α of the photonic radiation in the considered heterostructure (Figs. 1, 2, 4) for $h\nu > 2.70$ eV.

Taking into account the previously considered model (Fig. 3) the electron from the state $K' = 0$ has a greater energy than that corresponding to the inferior minimum value with $K_2 = K_{\min}$ and that is why we think that after a very short time, $\tau \approx 10^{-13}$ s, this will pass in the minimum energy state K_2 absorbing or giving up a phonon-transition (2) (Fig. 3).

Another mechanism of the interface photoconduction is that by which, as a result of a photon absorption, the electron is passing through a vertical transition (3) (Fig. 3), from a low state in the valence zone $K_1 \neq 0$ to the state $K_2 = K_{\min}$ in the conduction zone. The hole from the valence zone makes the transition (4) in the state with $K_1 = 0$ near the superior edge of the interface valence zone by the absorption or emission of a photon-relations (4) (Figs. 1, 2, 4).

Consequently, we assume that in the conditions in which $h\nu = \hbar\omega > \phi_B$, where $\phi_B = \varepsilon_f - \varepsilon_i$ (relations (4)) the considered indirect transitions will determine a maximum absorption in the region of the high wave lengths as compared with the absorption edge of the metal/semiconductor interface, a fact which is in agreement, experimentally, for $h\nu > 0.95$ eV (Figs. 1, 2, 4).

The absorption coefficient α corresponding to the considered indirect transition will be made up of two terms corresponding to the absorption α_a and to the emission α_e of the phonon, respectively:

$$(7) \quad \alpha(h\nu) = \alpha_a(h\nu) \pm \alpha_e(h\nu),$$

where $\alpha(h\nu) = \alpha(\hbar\omega) = [\ln(I_0/I)]/l$, I_0 is the intensity of the incident radiation, l is the thickness of the absorption layer at the interface, while I represents the intensity of the radiation corresponding to the l thickness.

In the relation (7), the absorption coefficient $\alpha_a(h\nu) = \alpha_a(\hbar\omega)$ corresponding to transition with phonons absorption is proportional to the probability $P(N_{\text{phon}})$ of interacting with the phonons, which is a function of the phonons number N_{phon} with the energy $\varepsilon_{\text{phon}}$ from the interface structural lattice

$$P(N_{\text{phon}}) \sim N_{\text{phon}},$$

$$N_{\text{phon}} = \left(e^{\varepsilon_{\text{phon}}/kT} - 1 \right)^{-1}$$

being the Planck distribution function [2]. Under the conditions the tests have been done, for $h\nu > \phi_B - \varepsilon_{\text{phon}}$, $\phi_B = \varepsilon_f - \varepsilon_i$, - see relations (4) - the absorption coefficient $\alpha(h\nu)$ corresponding to the phonons absorption has, for this case, the form

$$(8) \quad \alpha_a(h\nu) \approx \frac{(\hbar\omega - \phi_B + \varepsilon_{\text{phon}})^2}{e^{\varepsilon_{\text{phon}}/kT} - 1}.$$

The emission probability of the phonon is proportional to $N_{\text{phon}} + 1$, so that under the condition in which $h\nu > \phi_B + \varepsilon_{\text{phon}}$, the absorption coefficient $\alpha_e(h\nu)$ for the transition with the phonon emission will have the form

$$(9) \quad \alpha_e(h\nu) \approx \frac{(\hbar\omega - \phi_B - \varepsilon_{\text{phon}})^2}{1 - e^{\varepsilon_{\text{phon}}/kT}}.$$

As our experiments (at room temperature $T \approx 300^\circ \text{ K}$) have been done under the conditions for which $h\nu = \hbar\omega \gg \phi_B$ and, thus, for $h\nu = \hbar\omega > \phi_B + \varepsilon_{\text{phon}}$, we consider as being possible both the emission and those with the absorption of the phonons, the absorption coefficient being represented as the sum from the relation (7), $\alpha = \alpha_a + \alpha_e$.

Thus, the dependence $\sqrt{\alpha} = f(h\nu)$ (Fig. 4) (the experiments being done at the room temperature, $T \approx 300^\circ \text{ K}$) is characterized by two linear portions corresponding to these two terms from the relation (7): α_e , the absorption coefficient for the phonon emission and α_a , the absorption coefficient for the phonon absorption, respectively. The straight line (a) (Fig. 4) corresponding to the dependence $\sqrt{\alpha_e} = f_1(h\nu)$, intersects the abscissa in the point $h\nu = \phi_B + \varepsilon_{\text{phon}} \approx 3.02 \text{ eV}$ while the straight line (b) characterizing the dependence $\sqrt{\alpha_a} = f_2(h\nu)$ intersects the energy axis in the point $h\nu = \phi_B - \varepsilon_{\text{phon}} \approx 0.95 \text{ eV}$ (Fig. 4). Thus, the magnitude of the segment comprised between the intersection points of the straight lines (a) and (b) with the abscissa $h\nu$ (for $T \approx 300^\circ \text{ K}$) is $2\varepsilon_{\text{phon}} = 2.07 \text{ eV}$, and the middle point of this values agree with our experimental data at the plotting of both the voltamperic and of the photoelectric characteristics, respectively ($h\nu = \varepsilon_f - \varepsilon_i = \phi_B \approx 1.03 \text{ eV}$).

The experimental results showed the photovoltaic properties of this kind of mixed photoelement are highly influenced by the duration of the alternative exposition to light and dark, the illumination period playing a special part in the processes occurring at the Al/Chl and Chl/Cu interfaces (Fig. 5).

On taking into account the interface lattice structure of the $\text{SnO}_x/\text{Cd:Na-Cu}_x\text{S: Cu-Al-Chl/Cu}$ multijunction, characterized by the existence of some different types of ions (Cu, Mg, Na) we think that the scattering mechanism on phonons during the photonic absorption, is also determined by the existence of the dipole moments. Thus, we shall consider the constituent compounds of the multijunction as a totality of electric dipoles which absorb the photonic energy and we shall assume that the energetic transfer between the photosynthetic pigment molecules is done by the dipolar transition resonance phenomena; the directioning of the energy flow we have experimentally determined, is done from small to high wave lengths in the spectrum - (a) shift to red (Fig. 2). These reasons being taken into account, we could explain the fact that the strongest photonic interaction experimentally determined, is produced in the region of higher lengths where the photonic radiation frequency is comparable with the dipole oscillations ones the process being accomplished with the emission of an optical phonon flow (Figs. 2, 4).

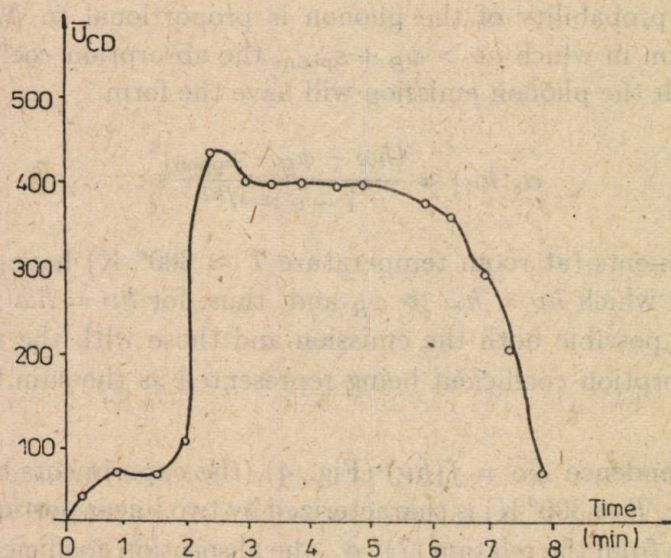


Fig. 5 — Variation of photovoltage at room temperature vs. illumination time.

Consequently, we think that the photonic absorption on the dipolar vibrations of the interface lattice, will determine a decrease in the photovoltaic efficiency, the emission process of the optical photon flow significantly influencing (under the conditions in which the experiments have been done at $T \approx 300^\circ \text{K}$) the value of the relation time τ_f of the photo-answer (Fig. 5).

At the same time, the experimental tests accomplished on the Al-Chl-Cu heterostructure, led to the conclusion that in the study of the photoelectric processes from the Al/Chl and Chl/Cu interfaces, the tunnelling and recombining effects, the contribution of the interface spatial load as well as the influence of the images forces on the height of potential barrier at the metal-semiconductor contact must be taken

into account. Thus, when determining the Schottky barrier height at the Al/Chl interface in the presence of an externally applied electric field, we consider that the correction introduced by the images forces (Schottky effect) should be taken into account.

On considering the presence of a Schottky-type barrier at the metal/chlorophyll interface, we have started to study the influence of external electric fields both on the energetic transfer and electron transfer mechanism through the photosynthetic membrane of the pigment molecules, as well as on the electric loading states from the Al/Chl and Chl/Cu interfaces.

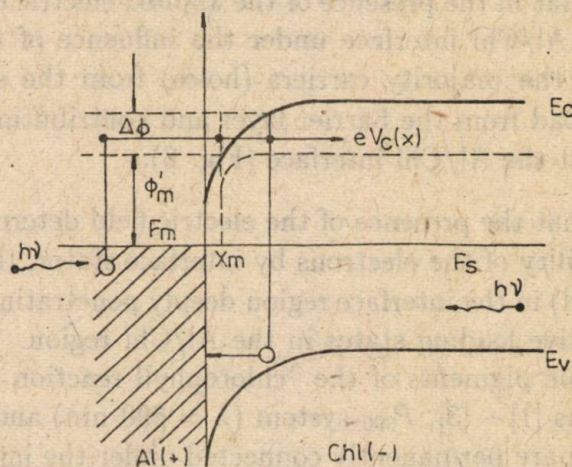


Fig. 6 — The barrier layer at Al/Chl interface.

Therefore, under the experiments condition, we could find out a slight increase of the photocurrent values for the samples subjected to external electric fields with intensity values ranging between 10^4 V/cm and 10^5 V/cm; $J_{SC} = 15...15.5$ nA/cm². This increase is explained both by the quantum tunnelling effect of the electrons through the Al/Chl barrier from Al through the Chl film (the electrons being scattered in the thin Cu layer at the Chl/Cu interface), and by the effect of the images forces at the Al/Chl interface, an effect which can determine a decrease in the Schottky barrier height. The inverse current presented during the tests a slight variation with the applied tension.

On resorting to analogical modelling, and on applying the Crowell and Sze theory, the tunnelling phenomena of the carriers (electrons) through the Al/Chl barrier, the recombining phenomenon of the carriers at the Al/Chl interface (due to the surface states) and the effect of the images forces at the metal-semiconductor interface [2], [4], have been considered for the determination of the barrier height. Thus, on taking into account the previously discussed facts, we have computed the decreases $\Delta\phi$ of the Schottky barrier height at the Al/Chl interface by using the

relation

$$(10) \quad \Delta\phi = \left(\frac{e^3 E}{4\pi\epsilon_0\epsilon_s} \right)^{\frac{1}{2}}.$$

We have found out a slight decrease of the barrier height, the mean values for $\Delta\phi$ (under the tests conditions) ranging between $\Delta\phi = 0.012...0.025$ eV (Fig. 6.)

In spite of the fact that this diminishing of the barrier height is small, we consider that this effect has an important influence on the passage of the current at the Al/Chl interface; thus, the behaviour of the photocurrent when an external electric current is applied is experimentally explained.

We also consider that in the presence of the applied electric current, the negative loading states at the Al/Chl interface under the influence of the images interface forces, would attract the majority carriers (holes) from the semiconductor, thus reducing the spatial load from the barrier layer and contributing to the decrease of the Schottky barrier at the Al/Chl interface (Fig. 6).

We also suppose that the presence of the electric field determines the increase of the tunnelling probability of the electrons by interface states, the wave functions of the metal electrons (Al) in the interface region deeply penetrating the semiconductor and introducing negative loading states in the Al/Chl region. Taking into account that the photosynthetic pigments of the "chlorophyll reaction center" type are organized in two systems [1] - [3], P_{690} system ($\lambda = 690$ nm) and P_{700} ($\lambda = 700$ nm), we consider that these are permanently connected under the influence of photos, by an electron carrying chain. Thus, as the result of high $h\nu$ energy absorption, the P_{700} system emits electrons on an intermediate level n_3 of unknown electronegative acceptor (Fig. 7)

The P_{700} system regains its lost electron by means of an intermediate compound (suggested to be on the form: enzyme+ vitamine C energetic level n'), the electron excitation levels of the pigment molecules depending on the wavelenght of the photon that enters the excitation resonance with the pigment molecule "reaction center", similarly with the optical pumping phenomenon produced in the LASER effect (Fig. 7).

At the same time, during the energy transfer from the chlorophyll molecule P_{690} to the molecule P_{700} , a photon excitation occurs in the electrons from the magnesium atom (Mg^{++} included in pigment structure), this excitation determining two of the electrons of the atom to populate the higher energy levels, thus leaving the chlorophyll molecule, so that the free electrons with a high energy level can be transported (by means of enzymes and vitamins) through the photosynthetic membrane of the pigment molecules. On taking all these facts into account, we assign to chlorophyll molecules a p -type semiconduction behaviour.

We also consider that both the electric field E and the ascorbic acid addition stimulate the protein and enzyme synthesis, resulting in the simplification of the elec-

tron transport through the photosynthetic membrane of the chlorophyll molecules, which renders evident a close correlation between the electric field, ascorbic acid, proteic metabolism and photoconduction mechanism in the photosynthetic pigment molecules.

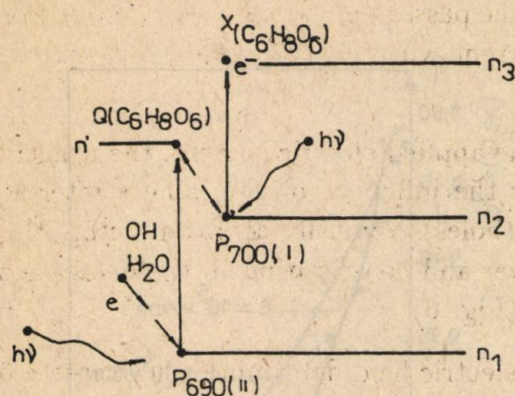


Fig. 7 — Scheme of photosynthetic electron transfer.

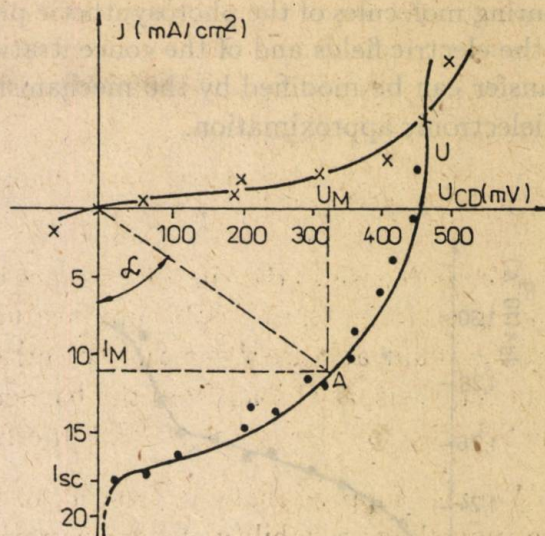


Fig. 8 — The $J - V$ characteristics for SnO_x/CdS , $\text{Na-Cu}_x\text{S}$, Cu-Al-Chl/Cu cell.

On taking into account the influence of the ascorbic acid on the electron transport through the photosynthetic membrane, we suppose that the Q and X intermediate factors from the P_{690} and P_{700} systems have the form: enzyme+ascorbic acid (Fig. 7).

The experimental results obtained so far, have made us come to the conclusion that both the impurities from I_a and I_b groups (Na, Li, Cu, K), and the photosynthetic pigments can improve the photoeffects at the $\text{Cd-Cu}_x\text{S}$ junction, by enhancing its photovoltages and photocurrents ($U_{CD} = 350 \dots 450 \text{ mV}$; $J_{SC} = 10 \dots 10.5 \text{ mA/cm}^2$) [1], [2].

We think that, under the influence of temperature ($100 \dots 150^\circ \text{C}$) the Na_2SO_4 and CuSO_4 thin films formed during the sputtering process give up the oxygen in the structural lattice of the heterojunction: oxygen diffuses and gets fixed at the $\text{CdS/Cu}_x\text{S}$ interface, thus determining the generation of a reactifying junction. We also consider that during this process a special part is played by the two physico-chemical parameters, temperature and concentration of the doping solutions, the optimum values that we have found for concentrations being 0.75% for Na_2SO_4 and 1.2% for CuSO_4 solutions, respectively.

We consider that in the photovoltaic behaviour of this kind of mixed multijunction, the microcrystalline chlorophyll thin films bring a special contribution. This pigment traps photons and passes them over to different energy levels, by molecular

induction or by inductive resonance [1], [2], the excitation levels of π electrons of the chlorophyll molecules depending on the wavelength of the photon entered the excitation resonance with the pigment molecule. Taking into account the obtained experimental results, we also consider that the electronic transfer between two neighbouring molecules of the photosynthetic pigment is conditioned by the presence both of the electric fields and of the concentration gradients of the dopants solutions; this transfer can be modified by the mechanism of the quanta-chemical tunnelling in an unielectronic approximation.

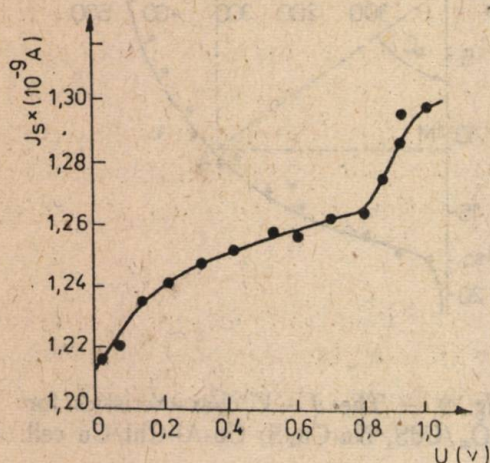


Fig. 9 — The $J - V$ characteristics for SnO_x/CdS , $\text{Na-Cu}_x\text{S}$, Cu-Al-Chl/Cu .

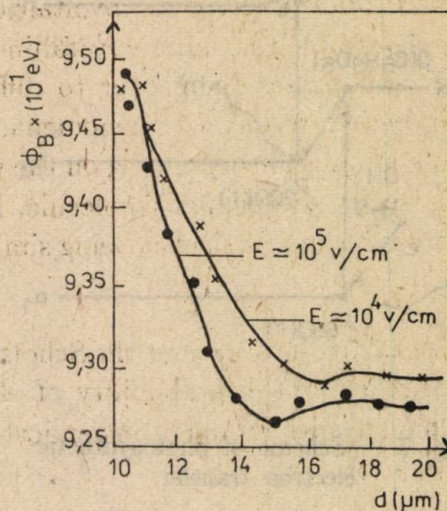


Fig. 10 — The $\phi_B = f(d)$ dependence for Al-Chl-Cu Schottky structure.

We also remark that, for this type of mixed multijunction, the SnO_x negative electrode being semitransparent, it is possible to illuminate the photoelement on its both faces (bilateral illumination), a fact which increases the charge carriers collection gain in the Cu_xS film of the heterostructure. In Fig. 8 we give the volt-ampere characteristic for such a mixed photoelement; here we can see an improvement of the maximum values of the open circuit voltage U_{CD} and short circuit current J_{SC} . The inverse current presented during the tests has a slight variation with the applied tension (Fig. 9). We have found out a slight decrease of the barrier height, the mean values for $\Delta\phi_B$ (under the tests conditions) ranging between $\Delta\phi = 0.012 \dots 0.025$ eV [1], [2] (Fig. 6). On considering the negative spatial charge region (x_d) as being included in the surface layer [1], [2] at the Al/Chl interface, $x_d \leq x \leq d$, we have studied the influence of the pigment layer thickness d on the Schottky barrier height ϕ_B (Fig. 10). The experimental plotting of the curve $\phi_B = f(d)$ has underlined a diminishing of the barrier at the same time with the increase of the pigment layer thickness d at the Al/Chl, Chl/Cu interface. From the diagram (Fig. 5) there results an optimum value for d which ranges between $14 \dots 16 \mu\text{m}$, therefore $d = 15 \mu\text{m}$ corresponding to the maximum value for $\Delta\phi_B$ in the interval $0.020 \dots 0.025$ eV (Figs. 9, 10).

3. Final Conclusion

1. Our experimental results have made us come to the conclusion that both the photosynthetic pigments and the impurities of the I_a and I_b groups can lead to the improvement of the photoeffects at the CdS-Cu_xS interface, by increasing their photovoltages and photocurrents: $U_{CD} = 350...575$ mV and $J_{SC} = 10.5...15$ mA/cm².

2. We think that the photovoltaic characteristics of this mixed multijunction are mainly determined by microcrystalline chlorophyll thin films. This pigment traps photons and passes them over to different energy levels, by molecular induction or by inductive resonance phenomena, the electrons excitation energy levels of the chlorophyll molecule depending on the wavelength of the atom entered the excitation resonance with the pigment molecule. By using the analogical modelling, we have considered this mechanism as being similar to the optical pumping met at the LASER effect.

3. On taking into account the Schottky barrier at the metal/ chlorophyll interface, we have revealed the probability of electronic transfer between two neighbouring pigment molecules by π -quanto-chemical tunnelling mechanism in the unielectronical approximation.

4. Thus, on resorting to the analogical modelling and on using the experimental results obtained at the plotting of the photovoltaic and photoelectric characteristics, we have obtained for the Schottky barrier height mean values ranging in the interval 0.95...1.15 eV; this result explains, to a great extent, from an experimental view point, the specific behaviour of the photocurrent and photopotential during our tests. The experimental results obtained at the plotting of the photoelectric characteristics both at the Al-Chl-Cu hererostructure and at the mixed multijunction we have accomplished, have also rendered evident both the presence of a Schottky-type barrier at the Al/Chl interface and the modification of the photocurrent and photopotential behaviour under the influence of the external electric fields, as a result of the decrease of the Schottky barrier height; $\Delta\phi = 0.012...0.025$ eV in their presence. Under these conditions, we have noticed the fact that, instead of the saturation phenomenon of the inverse current, a slight variation of the current with the applied tension has been produced, the mean values at the Al-Chl-Cu heterostructure ranging between 15...15.5 nA/cm².

5. The experiments showed, as well, that the interdependence between the optimum values of the physico-chemical parameters, namely, temperature, concentration of the doping solutions and the electric field E , plays a determining part in the physical processes at the CdS/Cu, Al/ Chl and Chl/Cu interfaces, considering also the probability of the electronic transfer between the neighbouring pigment molecules by means of the π -quanto-chemical tunnelling mechanism.

6. We consider that the explanation of the mechanism of the energy transfer and electron transport through the photosynthetic membrane, as well as the study of the photoconductive properties of different forms of microcrystalline chlorophyll present a special fundamental and practical interest for the utilisation of the photosynthetic pigment films in photovoltaic devices based on anorganic semiconductor materials.

7. The obtained experimental results have also emphasized the fact that the strongest photonic interaction takes place during the optical absorption process in the region of the highest wave lengths, $h\nu > 0.90$ eV, where probably of the photonic radiation frequency is comparable with the dipole oscillations one from the structural lattice; the process is solved by the emission of an optical phonon flow $2\epsilon_{\text{phon}} \approx \approx 2.07$ eV, which influences, up to a certain degree, the photovoltaic efficiency of this type of multilayer heterojunction.

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REFERENCES

1. Gherghel M., Melinte S., *Renewable Energy Sources*. Vol. 1, Research Center Cairo, Egypt and Wayne State University, U.S.A., pp. 179-196, 1990.
2. Gherghel M., Melinte S., *Project Hydrogen '91*. Conference Proceedings, American Academy of Science, pp. 13-35, 1991.
3. Gherghel M., *Bul. Inst. Polit. Iași*, **XXXVI (XL)**, 1-4, s. I, 101-106 (1994).
4. Gherghel M., *Solid State Phenomena*, **32-33**, 495-510 (1993).

ASUPRA UNOR PROPRIETĂȚI ELECTROOPTICE DE INTERFAȚĂ LA HETEROSTRUCTURI SEMICONDUCTOARE MIXTE CU PIGMENT FOTOSINTETIC

II. Fenomenul de absorbție optică la interfețe în heterojuncțiuni tip multistrat

(Rezumat)

Se prezintă rezultatele experimentale privind realizarea unui sistem fotovoltaic din straturi subțiri fotoconductoare tip multijoncțiune, pe bază de $\text{CdS-Cu}_x\text{S}$ și pigmenți fotosintetici. Sunt prezentate atât concluziile privind studiul comportării fotoelectrice și fotovoltaice a acestui sistem cât și cele privitoare la procesele electro-optice de interfață "metal-semiconductor", puse în evidență la acest tip de heterostructură. Cercetările experimentale privind studiul absorbției optice și a caracteristicilor fotovoltaice au evidențiat faptul că structura Al-CHl-Cu prezintă proprietăți fotoelectrice considerabile care au determinat o îmbunătățire substanțială a fotoefectelor și a eficienței de conversie la acest tip de dispozitiv. De asemenea, în cadrul explicării mecanismului transferului de purtători de sarcină în regiunile fotoactive, precum și al fenomenului de absorbție optică la interfețele structurii, s-au găsit similitudini între comportarea fotoelectrică a acestui tip de sistem fotovoltaic și comportarea electrooptică specifică unei structuri optoelectronice cu transfer (cuplare) de sarcini.

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Secția I
MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

RECENZII

BOOK REVIEWS

V. N. AFANAS'EV, V.B. KOLMANOVSKII and V. R. NOSOV, **Mathematical Theory of Control Systems Design**. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996, 668 pp., ISBN 0-7923-3724-7.

This volume deals with many interesting topics in the *Field of Control Theory and its Applications*. It is divided into an introduction and four parts. Each chapter concludes with a short overview of the main results and formulas, and each part ends with an exercise section.

Part One treats the fundamentals of *Modern Stability Theory*, while Part Two is devoted to the *Optimal Control of Deterministic Systems*. Part Three is concerned with problems of the *Control of Systems under Random Disturbances of their Parameters*, and Part Four provides an outline of *Modern Numerical Methods of Control Theory*. The many examples included serve to illustrate the main assertions and algorithms, as well as to teach the reader the skills needed to construct models of relevant phenomena, to design nonlinear control systems, to explain the qualitative differences between various classes of control systems, and to apply in the investigation of particular systems.

This book will be valuable to both graduate and postgraduate students in a variety of disciplines such as applied mathematics, mechanics, engineering, automation and cybernetics.

MARTIN COSTABEL (Ed.), **Boundary Value Problems and Integral Equations in Nonsmooth Domains**. Marcel Dekker, Inc., New York, U.S.A., 1995, 299 pp., ISBN 0-8247-9320-X.

Based on the "International Conference on Boundary Value Problems and Integral Equations in Nonsmooth Domains" held recently at the Centre International de Rencontres Mathématiques (CIRM), Luminy-France, this unique reference contains strongly interrelated, refereed papers that detail the latest findings in the fields of *Nonsmooth Domains and Corner Singularities, Examining Two-dimensional Polygonal or Lipschitz Domains, Three-dimensional Polyhedral Corners and Edges, and Conical Points in any Dimension*.

Presenting the newest results - some of them obtained after the Luminy conference took place, *Boundary Value Problems and Integral Equations in Nonsmooth Domains* investigates current research in the singularities of nonlinear boundary value problems, boundary layers, Stokes, Navier-Stokes, Lamé, Maxwell and Helmholtz equations, isotropic and anisotropic elasticity theory, the computation of singular exponents for two-dimensional and three-dimensional corners, edges with variable angles and branching singularities, adaptive finite element and boundary element methods, wavelet bases and other remarkable topics.

MICHAEL I. GIL, **Norm Estimations for Operator-Valued Functions and Applications**. Marcel Dekker, Inc., New York, U.S.A., 1995, 355 pp., ISBN 0-8247-9609-8.

Providing valuable new tools for specialists in functional analysis and stability theory, this state-of-the-art reference presents a systematic exposition of *Estimations for Norms of Operator-Valued Functions* and applies the estimates to *Spectrum Perturbations of Linear Operators and Stability Theory*.

Demonstrating a novel approach to spectrum perturbations the monograph considers among others a common procedure for the stability analysis of various classes of equations, extends the well-known spectrum perturbation result for self-adjoint operators to quasi-Hermitian operators, examines spectrum perturbations of operators on a tensor product of Hilbert spaces, covers systems of ordinary differential equations, deals with retarded systems, studies the absolute stability of systems of Volterra equations and emphasizes semilinear evolution equations.

Norm Estimations for Operator-Valued Functions and Applications is an ideal resource for mathematicians specializing in functional analysis, stability theory and control systems theory, mathematical biologists, circuit design engineers and graduate students in these disciplines.

STEPHEN BOYD, LAURENT EL GHAONI, ERIC FERON and VENKATARAMANAN BALAKRISHNAN, **Linear Matrix Inequalities in System and Control Theory**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, U.S.A., 1994, 193 pp., ISBN 0-89871-334-X.

In this book the authors reduce a wide *Variety of Problems Arising in System and Control Theory* to a handful of convex and quasiconvex optimization problems that involve linear matrix inequalities. These optimization problems can be solved using recently developed numerical algorithms that not only are polynomial-time but also work very well in practice; the reduction therefore can be considered a solution to the original problems. This book *opens up an important new research area* in which convex optimization is combined with system and control theory, resulting in the solution of a large number of previously unsolved problems.

C o n t e n t s: Preface; Ch. 1, Introduction: Overview; A Brief History of LMIs in Control Theory; Notes on the Style of the Book; Origin of the Book; Ch. 2, Some Standard Problems Involving LMIs: Linear Matrix Inequalities; Some Standard Problems; Ellipsoid Algorithm; Interior-Point Methods; Strict and Nonstrict LMIs; Miscellaneous Results on Matrix Inequalities; Some LMI Problems with Analytic Solutions; Ch. 3, Some Matrix Problems: Minimizing Condition Number by Scaling; Minimizing Condition Number of a Positive-Definite Matrix; Minimizing Norm by Scaling; Rescaling a Matrix Positive-Definite; Matrix Completion Problems; Quadratic Approximation of a Polytopic Norm; Ellipsoidal Approximation; Ch. 4, Linear Differential Inclusions: Differential Inclusions; Some Specific LDIs; Nonlinear System Analysis via LDIs; Ch. 5, Analysis of LDIs: State Properties; Quadratic Stability; Invariant Ellipsoids; Ch. 6, Analysis of LDIs: Input/Output Properties; Input-to-State Properties; State-to-Output Properties; Input-to-Output Properties; Observer-Based Controllers for Nonlinear Systems; Ch. 8, Lur'e and Multiplier Methods: Analysis of Lur'e Systems; Integral Quadratic Constraints; Multiplier for Systems with Unknown Parameters; Ch. 9, Systems with Multiplicative Noise: Analysis of Systems with Multiplicative Noise; State-Feedback Synthesis; Ch. 10, Miscellaneous Problems: Optimization over an Affine Family of Linear Systems; Analysis of Systems with LTI Perturbations; Positive Orthant Stabilizability; Linear Systems with Delays; Interpolation Problems; The Inverse Problem of Optimal Control; Multi-Criterion LQG; Nonconvex Multi-Criterion Quadratic Problems; System Realization Problems; Notation, List of Acronyms, Bibliography; Index.

The book identifies a handful of standard optimization problems that are general (a wide variety of problems from system and control theory can be reduced to them) as well as specific (specialized numerical algorithms can be devised for them).

The book catalogs a diverse list of problems in system and control theory that can be reduced to the standard optimization problems. Problems considered are analysis and state-feedback design for uncertain systems, matrix analysis problems and many others.

URI M. ASCHER, ROBERT M. M. MATTHEIJ and ROBERT D. RUSSELL, **Numerical Solution of Boundary Value Problems for Ordinary Differential Equations**. Classics in Applied Mathematics 13, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, U.S.A., 1995, 592 pp., ISBN 0-89871-854-4.

This book is the most comprehensive, up-to-date account of the popular numerical methods for solving boundary value problems in ordinary differential equations. It aims at a thorough understanding of the field by giving an indepth analysis of the numerical methods by using decoupling principles. Numerous exercises and real-world examples are used throughout to demonstrate the methods and the theory. Although first published in 1988, this republication remains the most comprehensive theoretical coverage of the subject matter, not available elsewhere is one volume.

Many problems, arising in a wide variety of application areas, give rise to mathematical models which form boundary value problems for ordinary differential equations. These problems rarely have a closed form solution, and computer simulation is typically used to obtain their approximate solution. This book discusses methods to carry out such computer simulations in a robust, efficient and reliable manner.

C o n t e n t s: List of Examples; Preface; Ch. 1, Introduction: *Boundary Value Problems for Ordinary Differential Equations; Boundary Value Problems in Applications*; Ch. 2, Review of Numerical Analysis and Mathematical Background: *Errors in Computation; Numerical Linear Algebra; Nonlinear Equations; Polynomial Interpolation; Piecewise Polynomials, or Splines; Numerical Quadrature; Initial Value Ordinary Differential Equations; Differential Operators and Their Discretizations*; Ch. 3, Theory of Ordinary Differential Equations: *Existence and Uniqueness Results; Green's Functions; Stability of Initial Value Problems; Conditioning of Boundary Value Problems*; Ch. 4, Initial Value Methods: *Introduction; Shooting; Superposition and Reduced Superposition; Multiple Shooting for Linear Problems; Marching Techniques for Multiple Shooting; The Riccati Method; Nonlinear Problems*; Ch. 5, Finite Difference Methods: *Introduction; Consistency, Stability and Convergence; Higher-Order One-Step Schemes; Collocation Theory; Acceleration Techniques; Higher-Order ODEs; Finite Element Methods*; Ch. 6, Decoupling: *Decomposition of the ODE; Decoupling of One-Step Recursions; Practical Aspects of Consistency; Closure and its Implications*; Ch. 7, Solving Linear Equations: *General Staircase Matrices and Condensation; Algorithms for the Separated BC Case; Stability for Block Methods; Decomposition in the Nonseparated BC Case; Solution in More General Cases*; Ch 8, Solving Nonlinear Equations: *Improving the Local Convergence of Newton's Method; Reducing the Cost of the Newton Iteration; Finding a Good Initial Guess; Further Remarks on Discrete Nonlinear BVPS*; Ch. 9., Mesh Selection: *Introduction; Direct Methods; A Mesh Strategy for Collocation; Transformation Methods; General Considerations*; Ch. 10, Singular Perturbations: *Analytical Approaches; Numerical Approaches; Difference Methods; Initial Value Methods*; Ch. 11, Special Topics: *Reformulation of Problems in "Standard" Form; Generalized ODEs and Differential Algebraic Equations; Eigenvalue Problems; BVPs with Singularities; Infinite Intervals; Path Following, Singular Points and Bifurcation; Highly Oscillatory Solutions; Functional Differential Equations; Method of Lines for PDEs; Multipoint Problems; On Code Design and Comparison; Appendix A, A Multiple Shooting Code; Appendix B, A Collocation Code; References; Bibliography; Index.*

Graduate and senior undergraduate students in applied mathematics, computer science and other areas in science and engineering, as well as researches in mathematical methods and others who use mathematical models in areas such as earth sciences, physics and mechanical engineering will benefit from reading this book. This text is recommended for a graduate course on boundary value problems or a senior undergraduate course on initial value problems or numerical ordinary differential equations.

AKE BJÖRCK, **Numerical Methods for Least Squares Problems**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, U.S.A., 1996, 408 pp., ISBN 0-89871-360-9.

The method of least squares was discovered by Gauss in 1795. It has since become the principal tool for reducing the influence of errors when fitting models to given observations. Today, applications of least squares arise in a great number of scientific areas, such as statistics, geodetics, signal processing and control.

In the last 20 years there has been a great increase in the capacity for automatic data capturing and computing. Least squares problems of large size are now routinely solved. Tremendous progress has been made in numerical methods for least squares problems, in particular for generalized and modified least squares problems and direct and iterative methods for sparse problems. Until now there has not been a monograph that covers the full spectrum of relevant problems and methods in least squares.

This volume gives an in-depth treatment of topics such as *Methods for Sparse Least Squares Problems*, *Iterative Methods*, *Modified Least Squares*, *Weighted Problems*, and *Constrained and Regularized Problems*. The more than 800 references provide a comprehensive survey of the available literature on the subject.

Mathematicians working in numerical linear algebra, computational scientists and engineers, statisticians and electrical engineers will find the book of interest. The book can also be used in upper-level undergraduate and beginning graduate courses in scientific computing and applied sciences.

G. W. STEWART, **Afternotes on Numerical Analysis**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, U.S.A., 1996, X+200 pp., ISBN 0-89871-362-5.

This book presents the central ideas of *Modern Numerical Analysis* in a vivid and straightforward fashion with a minimum of fuss and formality. Stewart designed this volume, while teaching a upperdivision course in introductory numerical analysis. To clarify what he was teaching, he wrote down each lecture immediately after it was given. The result reflects the wit, insight, and verbal craftsmanship which are hallmarks of the author.

Simple examples are used to introduce each topic, then the author quickly moves on to the discussion of important methods and techniques. With its rich mixture of graphs and code segments, the book provides insights and advice that help the reader avoid the many pitfalls in numerical computation that can easily trap an unwary beginner.

C o n t e n t s: **Part One, Nonlinear Equations:** Lecture 1, *By the Dawn's Early Light; Interval Bisection; Relative Error*; Lecture 2, *Newton's Method; Reciprocals and Square Roots; Local Convergence Analysis; Slow Death*; Lecture 3, *A Quasi Newton Method; Rates of Convergence; Iterating for a Fixed Point; Multiple Zeros; Ending with a Proposition*; Lecture 4, *The Secant Method; Convergence; Rate of Convergence; Multipoint Methods; Muller's Method; The Linear-Fractional Method*; Lecture 5, *A Hybrid Method; Errors, Accuracy, and Condition Numbers*. **Part Two, Computer Arithmetic:** Lecture 6, *Floating-Point Numbers; Overflow and Underflow; Rounding Error; Floating-point Arithmetic*; Lecture 7, *Computing Sums; Backward Error Analysis; Perturbation Analysis; Cheap and Chippy Chopping*; Lecture 8, *Cancellation; The Quadratic Equation; That Fatal Bit of Rounding Error; Envoi*. **Part Three, Linear Equations:** Lecture 9, *Matrices, Vectors and Scalars; Operations with Matrices; Rank-One Matrices; Partitioned Matrices*; Lecture 10, *Theory of Linear Systems; Computational Generalities; Triangular Systems; Operation Counts*; Lecture 11, *Memory Considerations; Row Oriented Algorithms; A Column Oriented Algorithm; General Observations on Row and Column Orientation; Basic Linear Algebra Subprograms*; Lecture 12, *Positive Definite Matrices; The Cholesky Decomposition; Economics*; Lecture 13, *Inner-Product Form of the Cholesky Algorithm; Gaussian Elimination*; Lecture 14, *Pivoting; BLAS; Upper Hessenberg and Tridiagonal Systems*; Lecture 15, *Vector Norms; Matrix Norms; Relative Error; Sensitivity of Linear Systems*; Lecture 16, *The Condition of Linear Systems; Artificial III Conditioning; Rounding Error and Gaussian Elimination; Comments on the Analysis*; Lecture 17, *The Wonderful Residual; A Project; Introduction; More on Norms; The Wonderful Residual; Matrices with Known Condition; Invert and Multiply; Cramer's Rule; Submission*. **Part Four, Polynomial Interpolation:** Lecture 18, *Quadratic Interpolation; Shifting; Polynomial Interpolation; Lagrange Polynomials and Existence; Uniqueness*; Lecture 19, *Synthetic Division; The Newton Form of the Interpolant; Evaluation; Existence; Divided Differences*; Lecture 20, *Error in Interpolation; Error Bounds; Convergence; Chebyshev Points*. **Part Five, Numerical Integration and Differentiation:** Lecture 21, *Numerical Integration; Change of Intervals; The Trapezoidal Rule; The Composite Trapezoidal Rule; Newton-Cotes Formulas; Undetermined Coefficients and Simpson's Rule*; Lecture 22, *The Composite Simpson's Rule; Errors in Simpson's Rule*;

Weighting Functions; Gaussian Quadrature; Lecture 23, The Setting; Orthogonal Polynomials; Existence; Zeros of Orthogonal Polynomials; Gaussian Quadrature; Error and Convergence; Examples; Lecture 24, Numerical Differentiation and Integration; Formulas From Power Series; Limitations; Bibliography.

Written by a leading expert in numerical analysis, this book is certain to be the one favorite textbook as supplementary text for upper-division undergraduate and beginning graduate introductory numerical analysis courses.

* * **Proceedings of the Seventh Annual ACM-SIAM Symposium on Discrete Algorithms.** Proceedings in Applied Mathematics 81, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, U.S.A., 1996, 586 pp., ISBN 089871-366-8.

The papers in this volume were presented at the "Seventh Annual ACM-SIAM Symposium on Discrete Algorithms" held January 28-30, 1996 in Atlanta, Georgia. The *Symposium* was jointly sponsored by the *ACM Special Interest Group on Algorithm and Computation Theory* and the *SIAM Activity Group on Discrete Mathematics*.

The Program Committee met September 10-11, 1995 and selected the papers in this volume from 232 abstracts submitted in response to the call for papers. The selection was based on originality, quality and relevance to the study of algorithms.

The submissions were not refereed and many of them represent reports of continuing research. It is anticipated that most of these papers will appear in a more polished and complete form in scientific journals.





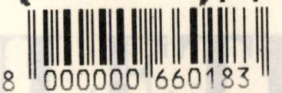
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Secția I

MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

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SOME ORDERED GROUPS ON SOME CARTESIAN PRODUCTS (III)

BY

GH. COSTOVICI

Abstract. Using 3-linear mappings, on cartesian products of four p.o. groups, one obtains new p.o. groups.

Key words: p.o. group, 3-linear mapping.

1. By means of 3-linear mappings, on some cartesian products of four partially ordered (p.o.) groups, we obtain some new p.o. groups and give some examples.

2. If G_1, G_2, G_3 and G are multiplicative groups, then the mapping $f : G_1 \times G_2 \times G_3 \rightarrow G$ is 3-linear $\Leftrightarrow$ the following are fulfilled:

$$(1) \quad \forall x, y \in G_1, \quad \forall b \in G_2 \quad \text{and} \quad \forall c \in G_3, \quad f(xy, b, c) = f(x, b, c)f(y, b, c),$$

$$(2) \quad \forall a \in G_1, \quad \forall x, y \in G_2 \quad \text{and} \quad \forall c \in G_3, \quad f(a, xy, c) = f(a, x, c)f(a, y, c),$$

$$(3) \quad \forall a \in G_1, \quad \forall b \in G_2 \quad \text{and} \quad \forall x, y \in G_3, \quad f(a, b, xy) = f(a, b, x)f(a, b, y).$$

If $f : G_1 \times G_2 \times G_3 \rightarrow G$ is a 3-linear mapping then the following take place:

$$(4) \quad \forall x \in G_1, \quad \forall y \in G_2, \quad \forall z \in G_3, \quad f(x, y, 1) = 1, \quad f(x, 1, z) = 1, \quad f(1, y, z) = 1$$

for $f(x, y, 1) = f(x, y, 11) = f(x, y, 1)f(x, y, 1)$ and similarly.

Theorem (T). I. If $f : G_1 \times G_2 \times G_3 \rightarrow G$ is a 3-linear mapping and G is commutative, then $H = G_1 \times G_2 \times G_3 \times G$ becomes a group by the operation $(a, b, c, d) \circ (x, y, z, t) = (ax, by, cz, dt f(a, y, z) f(x, b, z) f(x, y, c) f(a, b, z) f(a, y, c) f(x, b, c))$.

II. If besides the hypotheses of I, G_i $i = \overline{1, 3}$ and G are p.o. groups, then $(H, \circ)$ becomes a p.o. group by the lexicographic order that is with the positive cone $P = \{(x, y, z, t) \in H; \quad x > 1 \text{ or } (x = 1 \text{ and } y > 1) \text{ or } (x = 1, y = 1 \text{ and } z > 1) \text{ or } (x = 1, y = 1, z = 1 \text{ and } t \geq 1)\}$.

P r o o f of I. $[(a, b, c, d) \circ (x, y, z, t)] \circ (m, n, p, q) = (ax, by, cz, dt f(a, y, z) f(x, b, z) f(x, y, c) f(a, b, z) f(a, y, c) f(x, b, c)) \circ (m, n, p, q) = (axm, byn, czp, dt f(a, y, z) f(x, b, z) f(x, y, c) f(a, b, z) f(a, y, c) f(x, b, c) q f(ax, n, p) f(m, by, p) f(m, n, cz) f(ax, by, p) f(ax, n, cz) f(m, by, cz) = (axm, byn, czp, dt f(a, y, z) f(x, b, z) f(x, y, c) f(a, b, z) f(a, y, c) f(x, b, c) q f(a, n, p) f(x, n, p) f(m, b, p) f(m, y, p) f(m, n, c) f(m, n, z) f(a, b, p) f(a, y, p) f(x, b, p) f(x, y, p) f(a, n, c) f(a, n, z) f(x, n, c) f(x, n, z) f(m, b, c) f(m, b, z) f(m, y, c) f(m, y, z))$.

$(a, b, c, d) \circ [(x, y, z, t) \circ (m, n, p, q)] = (a, b, c, d) \circ (xm, yn, zp, tq f(x, n, p) f(m, y, p) f(m, n, z) f(x, y, p) f(x, n, z) f(m, y, z)) = (axm, byn, czp, dt q f(x, n, p) f(m, y, p) f(m, n, z) f(x, y, p) f(x, n, z) f(m, y, z) f(a, yn, zp) f(xm, b, zp) f(xm, yn, c) f(a, b, zp) f(a, yn, c) f(xm, b, c)) = (axm, byn, czp, dt q f(x, n, p) f(m, y, p) f(m, n, z) f(x, y, p) f(x, n, z) f(m, y, z) f(a, y, z) f(a, y, p) f(a, n, z) f(a, n, p) f(x, b, z) f(x, b, p) f(m, b, z) f(m, b, p) f(x, y, c) f(x, n, c) f(m, y, c) f(m, n, c) f(a, b, z) f(a, b, p) f(a, y, c) f(a, n, c) f(x, b, c) f(m, b, c))$.

G being commutative, the operation $\circ$ is associative.

$\exists (1, 1, 1, 1) \in H$ so that $\forall (x, y, z, t) \in H$ we have $(x, y, z, t) \circ (1, 1, 1, 1) = (x, y, z, t)$ by (4). If $(x, y, z, t) \in H$, $\exists (x^{-1}, y^{-1}, z^{-1}, t^{-1}) \in H$ so that $(x, y, z, t) \circ (x^{-1}, y^{-1}, z^{-1}, t^{-1}) = (1, 1, 1, 1 f(x, y^{-1}, z^{-1}) f(x^{-1}, y, z^{-1}) f(x^{-1}, y^{-1}, z) f(x, y, z^{-1}) f(x, y^{-1}, z) f(x^{-1}, y, z)) = (1, 1, 1, f(x, 1, z^{-1}) f(x^{-1}, y, 1) f(1, y^{-1}, z)) = (1, 1, 1, 1)$.

Proof of II. $P = \{(x, y, z, t) \in H; x > 1 \text{ or } (x = 1 \text{ and } y > 1) \text{ or } (x = 1, y = 1 \text{ and } z > 1) \text{ or } x = 1, y = 1, z = 1 \text{ and } t \geq 1\}$. We show that $P \cap P^{-1} = \{(1, 1, 1, 1)\}$.

Let $(x, y, z, t) \in P \cap P^{-1}$ be. Then $(x, y, z, t) \in P$ and $(x, y, z, t) = (p_1, p_2, p_3, p_4)^{-1} = (p_1^{-1}, p_2^{-1}, p_3^{-1}, p_4^{-1})$ with $(p_1, p_2, p_3, p_4) \in P$.

$x > 1$ and $x = p_1^{-1} < 1$, impossible. $x > 1$ and $x = p_1^{-1} = 1$, impossible. $x = 1, y > 1$ and $x = p_1^{-1} < 1$ impossible. $x = 1, y > 1, x = p_1^{-1} = 1$ and $y = p_2^{-1} < 1$, impossible. $x = 1, y > 1, x = p_1^{-1} = 1$ and $y = p_2^{-1} = 1$, impossible. $x = 1, y = 1, z > 1$, and $x = p_1^{-1} < 1$, impossible. $x = 1, y = 1, z > 1, x = p_1^{-1} = 1$ and $y = p_2^{-1} < 1$, impossible. $x = 1, y = 1, z > 1, x = p_1^{-1} = 1, y = p_2^{-1} = 1$ and $z = p_3^{-1} < 1$, impossible. $x = 1, y = 1, z > 1, x = p_1^{-1} = 1, y = p_2^{-1} = 1$ and $z = p_3^{-1} = 1$, impossible. $x = 1, y = 1, z = 1, t \geq 1, x = p_1^{-1} < 1$, impossible. $x = 1, y = 1, z = 1, t \geq 1, x = p_1^{-1} = 1, y = p_2^{-1} < 1$, impossible. $x = 1, y = 1, z = 1, t \geq 1, x = p_1^{-1} = 1, y = p_2^{-1} = 1, z = p_3^{-1} < 1$, impossible. $x = 1, y = 1, z = 1, t \geq 1, x = p_1^{-1} = 1, y = p_2^{-1} = 1, z = p_3^{-1} = 1, t = p_4^{-1} \leq 1 \Rightarrow x = 1, y = 1, z = 1, t = 1$.

We show that $P \circ P \subseteq P$.

Let $(a, b, c, d) \in P$ be $\Leftrightarrow a > 1$ or $(a = 1 \text{ and } b > 1)$ or $(a = 1, b = 1 \text{ and } c > 1)$ or $(a = 1, b = 1, c = 1 \text{ and } d \geq 1)$. Let $(x, y, z, t) \in P$ be $\Leftrightarrow x > 1$ or $(x = 1 \text{ and } y > 1)$ or $(x = 1, y = 1 \text{ and } z > 1)$ or $(x = 1, y = 1, z = 1 \text{ and } t \geq 1)$.

We have $r = (a, b, c, d) \circ (x, y, z, t) = (ax, by, cz, dt f(a, y, z) f(x, b, z) f(x, y, c) f(a, b, z) f(a, y, c) f(x, b, c))$.

$a > 1, x > 1 \Rightarrow ax > 1 \Rightarrow r \in P$. $a > 1, x = 1$ and $y > 1 \Rightarrow ax > 1 \Rightarrow r \in P$. $a = 1, b > 1$, and $x > 1, \Rightarrow ax > 1 \Rightarrow r \in P$. $a = 1, b > 1, x = 1, y > 1 \Rightarrow ax = 1$ and $by > 1 \Rightarrow r \in P$. $a = 1, b > 1, x = 1, y = 1$ and $z > 1 \Rightarrow ax = 1$ and $by > 1 \Rightarrow r \in P$. $a = 1, b = 1, c > 1$ and $x > 1 \Rightarrow ax > 1 \Rightarrow r \in P$. $a = 1, b = 1, c > 1, x = 1$ and $y > 1 \Rightarrow ax = 1$ and $by > 1 \Rightarrow r \in P$. $a = 1, b = 1, c > 1, x = 1, y = 1$ and $z > 1 \Rightarrow ax = 1, by = 1$ and $cz > 1 \Rightarrow r \in P$. $a = 1, b = 1, c > 1, x = 1, y = 1, z = 1$ and $t \geq 1 \Rightarrow ax = 1, by = 1$ and $cz > 1 \Rightarrow r \in P$. $a = 1, b = 1, c = 1, d \geq 1$ and $x > 1 \Rightarrow ax > 1 \Rightarrow r \in P$. $a = 1, b = 1, c = 1, d \geq 1, x = 1$ and $y > 1 \Rightarrow ax = 1, by > 1 \Rightarrow r \in P$. $a = 1, b = 1, c = 1, d \geq 1, x = 1, y = 1$, and $z > 1 \Rightarrow ax = 1, by = 1, cz > 1 \Rightarrow r \in P$. $a = 1, b = 1, c = 1, d \geq 1, x = 1, y = 1, z = 1$ and $t \geq 1 \Rightarrow ax = 1, by = 1, cz = 1, dt \geq 1, f(a, y, z) = f(x, b, z) = f(x, y, c) = f(a, b, z) = f(a, y, c) = f(x, b, c) = 1 \Rightarrow r \in P$.

We show that $\forall (x, y, z, t) \in H$ we have $(x, y, z, t) \circ P \circ (x, y, z, t)^{-1} \subseteq P$.

Let $(p_1, p_2, p_3, p_4) \in P$ be $\Leftrightarrow p_1 > 1$ or $(p_1 = 1$ and $p_2 > 1)$ or $(p_1 = 1, p_2 = 1$ and $p_3 > 1)$ or $(p_1 = 1, p_2 = 1, p_3 = 1$ and $p_4 \geq 1)$.

We have $r = (x, y, z, t) \circ (p_1, p_2, p_3, p_4) \circ (x, y, z, t)^{-1} = (xp_1, yp_2, zp_3, tp_4 f(x, p_2, p_3) f(p_1, y, p_3) f(p_1, p_2, z) f(x, y, p_3) f(x, p_2, z) f(p_1, y, z) \circ (x^{-1}, y^{-1}, z^{-1}, t^{-1}) = (xp_1 x^{-1}, yp_2 y^{-1}, zp_3 z^{-1}, tp_4 f(x, p_2, p_3) f(p_1, y, p_3) f(p_1, p_2, z) f(x, y, p_3) f(x, p_2, z) f(p_1, y, z) t^{-1} f(xp_1, y^{-1}, z^{-1}) f(x^{-1}, yp_2, z^{-1}) f(x^{-1}, y^{-1}, zp_3) f(xp_1, yp_2, z^{-1}) f(xp_1, y^{-1}, zp_3) f(x^{-1}, yp_2, zp_3)) = (xp_1 x^{-1}, yp_2 y^{-1}, zp_3 z^{-1}, p_4 f(x, p_2, p_3) f(p_1, y, p_3) f(p_1, p_2, z) f(x, y, p_3) f(x, p_2, z) f(p_1, y, z) f(x, y^{-1}, z^{-1}) f(p_1, y^{-1}, z^{-1}) f(x^{-1}, y, z^{-1}) f(x^{-1}, p_2, z^{-1}) f(x^{-1}, y^{-1}, z) f(x^{-1}, y^{-1}, p_3) f(x, y, z^{-1}) f(x, p_2, z^{-1}) f(p_1, y, z^{-1}) f(p_1, p_2, z^{-1}) f(x, y^{-1}, z) f(x, y^{-1}, p_3) f(p_1, y^{-1}, z) f(p_1, y^{-1}, p_3) f(x^{-1}, y, z) f(x^{-1}, y, p_3) f(x^{-1}, p_2, z) f(x^{-1}, p_2, p_3)) = (xp_1 x^{-1}, yp_2 y^{-1}, zp_3 z^{-1}, p_4 f(1, p_2, p_3) f(p_1, 1, p_3) f(p_1, p_2, 1) f(x, 1, p_3) f(x, p_2, 1) f(p_1, y, 1) f(x, 1, z^{-1}) f(p_1, y^{-1}, 1) f(x^{-1}, y, 1) f(x^{-1}, p_2, 1) f(1, y^{-1}, z) f(x^{-1}, 1, p_3) = (xp_1 x^{-1}, yp_2 y^{-1}, zp_3 z^{-1}, p_4)$.

Examples. 1. Let I be a ring (it can be nonassociative, noncommutative and without unity). The mapping $f : I \times I \times I \rightarrow I$ with $f(x, y, z) = (xy)z$, is 3-linear for $f(a + b, y, z) = ((a + b)y)z = (ay + by)z = (ay)z + (by)z = f(a, y, z) + f(b, y, z)$. We suppose that the additive group of I is a p.o. group. Then $H = I \times I \times I \times I$ becomes a p.o. group by the operation $(a, b, c, d) \circ (x, y, z, t) = (a + x, b + y, c + z, d + t + (ay)z + (xb)z + (xy)c + (ab)z + (ay)c + (xb)c)$ and by the lexicographic order.

2. Let G be a commutative p.o. group (multiplicative) and let k be a fixed element of G . On the set of integer numbers, $\mathbb{Z}$, we suppose the usual order and sum. The mapping $f : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \rightarrow G$ with $f(x, y, z) = k^{xyz}$ is a 3-linear for $f(x, y, a + b) =$

$= k^{xy(a+b)} = k^{xya+xyb} = k^{xya} k^{xyb} = f(x, y, a) f(x, y, b)$. Then $H = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times G$ becomes a p.o. group by the operation $(a, b, c, d) \circ (x, y, z, t) = (a+x, b+y, c+z, dtk^{ayz} k^{xbz} k^{xyc} k^{abz} k^{ayc} k^{xbc}) = (a+x, b+y, c+z, dtk^{ayz+xbz+xyc+abz+ayc+xbc})$ and by the lexicographic order.

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UNELE GRUPURI ORDONATE PE UNELE PRODUSE CARTEZIENE (III)

(Rezumat)

Utilizând aplicațiile triliniare, pe unele produse carteziane (de patru factori) de grupuri parțial ordonate, se obțin noi grupuri parțial ordonate și se dau unele exemple.

ON THE GEOMETRY OF CURVES IN $F^{(3)}$

BY

AUREL BEJANCU

Abstract. By using the Cartan connection we obtain the Frenet frame and the Frenet equations for a curve C in a three-dimensional Finsler manifold. Also we prove two theorems on the reduction of the codimension of immersion and give explicit formulas for curvature and torsion of C .

Key words: Frenet frame, Frenet equations, Finsler manifold, curvature, torsion.

0. Introduction

The purpose of the present paper is to initiate a study of geometry of curves in a three-dimensional Finsler manifold based on both the modern theory of Finsler connections (cf. M. Matsumoto [3]) and the theory of Finsler submanifolds via the vertical vector bundle (cf. A. Bejancu [1], [2]). A Frenet frame for a curve in a Finsler manifold has been first constructed by J. H. Taylor [5]. However, as far as we know, the matter was not further investigated, in spite of a tremendous research work on Finsler geometry in the last three decades.

In the first section we consider an m -dimensional Finsler manifold $F^{(m)}$ endowed with the Cartan connection FC and obtain the induced non-linear connection on the tangent bundle TC of a curve C in $F^{(m)}$. Also we show that the vertical covariant derivative does not provide any Frenet frame for C . Then by using the horizontal covariant derivative along C we obtain, in the second section, the Frenet frame and write down the Frenet equations for a curve C in $F^{(3)}$. Finally, in the last section we prove two theorems on the reduction of the codimension of C in $F^{(3)} = (\mathbb{R}^3, F)$ and derive formulas for both the curvature and torsion of C in a Minkowski space.

1. Curves in $F^{(m)}$

Let $F^{(m)} = (M, F)$ be a Finsler manifold, where M is a real m -dimensional manifold and F is the fundamental function of $F^{(m)}$ (cf. M. M a t s u m o t o [3], p. 101). Usually, F is not presumed to be smooth on the whole tangent bundle TM but on an open submanifold of TM .

Suppose C is a curve (1-dimensional submanifold) of M , locally given by the parametric equations

$$(1.1) \quad x^i = x^i(t), \quad t \in U,$$

where U is an open set of $\mathbb{R}$ and $(\dot{x}^1(t), \dots, \dot{x}^m(t)) \neq (0, \dots, 0)$ on U .

By $\dot{x}^i$, $\ddot{x}^i$ and $\dddot{x}^i$ we denote the first, second and third derivative of x^i with respect to t . We use the same notation for any other function. VTM will stand for the vertical vector bundle of M , i.e., $VTM = \text{Ker } d\pi$, where π is the natural projection of TM on M . By $\mathcal{F}(TC)$ and $\Gamma(E)$ we denote the algebra of smooth functions on TC and the $\mathcal{F}(TC)$ -module of smooth sections of a vector bundle E over TC . Throughout the paper we use the Einstein convention, that is, repeated indices with one upper and one lower index denotes summation over their range.

Taking into account that $F(x, y)$ is positive homogeneous of degree one with respect to y , the arc length parameter on C is defined by

$$(1.2) \quad s = \int_{t_0}^t F(x^i(t), \dot{x}^i(t)) dt.$$

From now on, in this section we suppose that C is locally given by the equations

$$(1.3) \quad x^i = x^i(s); \quad (x'^1(s), \dots, x'^m(s)) \neq (0, \dots, 0),$$

where $s \in (-\varepsilon, \varepsilon)$ is the above arc length parameter and primes denote derivatives with respect to s .

Denote by i the imbedding of C in M and consider the differential mapping $di: TC \rightarrow TM$. Locally, a point of TC with coordinates (s, v) is carried by di into a point of TM with coordinates $(x^i(s), y^i(s, v))$, where we set

$$(1.4) \quad y^i(s, v) = vx'^i(s).$$

By using (1.3) and (1.4) we deduce

$$(1.5) \quad \frac{\partial}{\partial s} = x'^i(s) \frac{\partial}{\partial x^i} + vx''^i(s) \frac{\partial}{\partial y^i},$$

and

$$(1.6) \quad \frac{\partial}{\partial v} = x'^i(s) \frac{\partial}{\partial y^i}.$$

Next, we recall that $\{\partial/\partial y^i\}$, $i \in \{1, \dots, m\}$ is a local basis for $\Gamma(VTM)$. By means of the fundamental function F of $F^{(m)}$ we obtain a Riemannian metric g on VTM given locally by (cf. A. B e j a n c u [2], p. 20)

$$(1.7) \quad g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}.$$

For any Finsler vector field X , that is, X is a smooth section of VTM , we denote by $\|X\|$ the norm of X which is defined by

$$\|X\| = \{g(X, X)\}^{1/2}.$$

Due to (1.6) we deduce that the vertical vector bundle VTC of C (locally spanned by $\partial/\partial v$) is a vector subbundle of $VTM|_{TC}$. On the other hand, from (1.2) we deduce

$$(1.8) \quad ds = F(x^i(t), \dot{x}^i(t))dt,$$

which implies

$$(1.9) \quad F(x^i(s), x'^i(s)) = 1.$$

As in general we have

$$(1.10) \quad F^2(x, y) = g_{ij}(x, y)y^i y^j,$$

from (1.9), (1.10) and (1.6) we deduce that $\partial/\partial v$ is a unit Finsler vector field, i.e., we have

$$(1.11) \quad \left\| \frac{\partial}{\partial v} \right\|^2 = g_{ij}(x(s), x'(s)) x'^i(s) x'^j(s) = 1.$$

Denote by $VTC^\perp$ the orthogonal complementary vector bundle to VTC in $VTM|_{TC}$ and call it as in theory of Finsler submanifold (cf. A. B e j a n c u [2], p. 47), the *Finsler normal bundle* of the imbedding of C in $F^{(m)}$. Hence we have the orthogonal decomposition

$$(1.12) \quad VTM|_{TC} = VTC \oplus VTC^\perp,$$

that is going to play an important role in construction of a Frenet frame for C .

In the present paper we consider the Finsler manifold $F^{(m)}$ endowed with Cartan connection $FC = (N_j^i, F_{jk}^i, C_{jk}^i)$ with local coefficients given by

$$(1.13) \quad N_j^i = \frac{\partial G^i}{\partial y^j}, \quad \text{where} \quad G^i = \frac{1}{4} g^{ih} \left(\frac{\partial^2 F^2}{\partial y^h \partial x^k} y^k - \frac{\partial F^2}{\partial x^h} \right),$$

$$(1.14) \quad F_{jk}^i = \frac{1}{2} g^{ih} \left(\frac{\delta g_{hj}}{\delta x^k} + \frac{\delta g_{hk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h} \right),$$

$$(1.15) \quad C_{jk}^i = \frac{1}{2} g^{ih} \frac{\partial g_{hj}}{\partial y^k},$$

where we use the differential operators

$$(1.16) \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}.$$

In another setting, the above Finsler connection is throughout as the pair $FC = (HTM, \nabla)$, where HTM is the complementary distribution to VTC in TTM locally represented by $\{\delta/\delta x^i\}$, $i \in \{1, \dots, m\}$ and ∇ is a metric linear connection on VTM locally expressed by (cf. A. B e j a n c u [2], p. 23)

$$(1.17) \quad \nabla_{\delta/\delta x^k} \frac{\partial}{\partial y^j} = F_{jk}^i \frac{\partial}{\partial y^i},$$

and

$$(1.18) \quad \nabla_{\partial/\partial y^k} \frac{\partial}{\partial y^j} = C_{jk}^i \frac{\partial}{\partial y^i}.$$

As F is not smooth on zero section of TM , we are dealing with geometric objects along C at points $(x(s), y(s, v))$ with $v \neq 0$. Hence we may suppose $v > 0$ for any point of the coordinate neighborhood involved in the study. From the homogeneity of F it follows that g_{ij} , g^{ij} and F_{jk}^i are positive homogeneous of degree zero, while G^i , N_j^i and C_{jk}^i are positive homogeneous of degree 2, 1 and -1 respectively. As a consequence we have

$$(1.19) \quad g_{ij}(x(s), vx'(s)) = g_{ij}(s),$$

$$(1.20) \quad x'^j(s)N_j^i(x(s), vx'(s)) = vx'^j(s)N_j^i(s) = 2vG^i(s),$$

$$(1.21) \quad x'^j(s)F_{jk}^i(x(s), vx'(s)) = x'^j(s)F_{jk}^i(s) = N_k^i(s),$$

$$(1.22) \quad x'^j(s)C_{jk}^i(x(s), vx'(s)) = 0,$$

and

$$(1.23) \quad C_{jk}^i(x(s), vx'(s)) = \frac{1}{v}C_{jk}^i(s).$$

Here and in the sequel, we use the vector notation $x(s)$ and $x'(s)$ to represent the vectors $(x^1(s), \dots, x^m(s))$ and $(x'^1(s), \dots, x'^m(s))$ respectively. Also, if $T_{kh\dots}^{ij\dots}$ are the components of a geometric object T at the point $(x(s), x'(s))$ we denote them by $T_{kh\dots}^{ij\dots}(s)$.

Now, by using (1.6), (1.18) and (1.22) we obtain

$$(1.24) \quad \nabla_{\partial/\partial v} \frac{\partial}{\partial v} = 0.$$

Therefore the vertical covariant derivative along C with respect to the Cartan connection does not provide any Frenet frame for C . Thus we have to proceed with horizontal covariant derivatives along C . To this end we need the induced non-linear connection HTC on TC which is locally represented by the vector field

$$(1.25) \quad \frac{\delta}{\delta s} = \frac{\partial}{\partial s} - f(s, v) \frac{\partial}{\partial v},$$

where f is obtained from the general formula given in A. Bejancu [1] for the local coefficients of the induced non-linear connection on a Finsler submanifold. By direct calculations using (1.20) we obtain

$$(1.26) \quad f(s, v) = vg_{ij}(s)(x''^i(s) + 2G^i(s))x'^j(s).$$

Then (1.24) and (1.25) yield

$$(1.27) \quad \nabla_{\delta/\delta s} \frac{\partial}{\partial v} = \nabla_{\partial/\partial s} \frac{\partial}{\partial v}.$$

Next, using (1.16) and (1.20) in (1.5) we derive

$$(1.28) \quad \frac{\partial}{\partial s} = x'^i(s) \frac{\delta}{\delta x^i} + v \left(x''^i(s) + 2G^i(s) \right) \frac{\partial}{\partial y^i},$$

at any point $(x(s), vx'(s)) \in TC$. Taking into account (1.28), (1.17), (1.18) and (1.20)–(1.22) we deduce

$$(1.29) \quad \nabla_{\partial/\partial s} \frac{\partial}{\partial v} = \left(x''^i(s) + 2G^i(s) \right) \frac{\partial}{\partial y^i}.$$

Finally, using (1.11) and taking into account that ∇ is a metric connection, we infer

$$(1.30) \quad g \left(\nabla_{\partial/\partial s} \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right) = 0.$$

Using (1.29) and (1.6) in (1.30) and comparing with (1.26) we obtain $f(s, v) = 0$. Hence we may state the following result.

Theorem 1.1. *The induced non-linear connection HTC on TC is locally spanned by $\partial/\partial s$ given by equivalent expressions from (1.5) and (1.28).*

2. The Frenet Frame for a Curve in $F^{(3)}$

Let C be a curve in a three-dimensional Finsler manifold $F^{(3)} = (M, F)$ given locally by the parametric equations

$$(2.1) \quad x^i = x^i(s); \quad (x'^1(s), x'^2(s), x'^3(s)) \neq (0, 0, 0), \quad i \in \{1, 2, 3\},$$

where s is the arc length parameter given by (1.2). Then from (1.30) we deduce that $\nabla_{\partial/\partial s}(\partial/\partial v) \in \Gamma(VTC^\perp)$ and therefore we may set

$$(2.2) \quad \nabla_{\partial/\partial s} \frac{\partial}{\partial v} = kn,$$

where $n \in \Gamma(VTC^\perp)$ is a unit Finsler vector field and

$$(2.3) \quad k = \left\| \nabla_{\partial/\partial s} \frac{\partial}{\partial v} \right\|.$$

Using (1.29) in (2.3) one obtains that

$$(2.4) \quad k = \left\{ g_{ij}(s) \left(x''^i(s) + 2G^i(s) \right) \left(x''^j(s) + 2G^j(s) \right) \right\}^{1/2}.$$

As in Riemannian geometry we call k the *curvature* of C .

Now we recall from H. Rund [4], p. 79 that C is a geodesic if and only if the following differential equations are fulfilled

$$(2.5) \quad x''^i(s) + 2G^i(s) = 0 \quad i \in \{1, 2, 3\}.$$

Thus (2.4) and (2.5) enable us to state the following result (compare with H. Rund [4], p. 152).

Theorem 2.1. *A curve C in $F^{(3)}$ is a geodesic if and only if the curvature k vanishes identically on C .*

If $k(s) \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$ we call the Finsler vector field n given by

$$(2.6) \quad n = \frac{1}{k(s)} \left(x''^i(s) + 2G^i(s) \right) \frac{\partial}{\partial y^i} = n^i(s) \frac{\partial}{\partial y^i},$$

the *principal normal Finsler vector field* of C . It is noteworthy that the local components n^i of n do not depend on v at any point $(x(s), y(s, v)) \in TC$.

Proposition 2.1. *The horizontal covariant derivative of the principal normal Finsler vector field with respect to $\partial/\partial s$ is given by*

$$(2.7) \quad \nabla_{\partial/\partial s} n = \left(\frac{dn^i}{ds} + n^j(s) B_j^i(s) \right) \frac{\partial}{\partial y^i},$$

where we set

$$(2.8) \quad B_j^i(s) = N_j^i(s) + \left(x''^k(s) + 2G^k(s) \right) C_{jk}^i(s).$$

Proof. The assertion follows by direct calculations using (1.28), (1.17), (1.18), (1.21) and (1.23).

By the above construction we have $\partial/\partial v$ and n as unit Finsler vector fields from the Frenet frame we are looking for. In order to obtain the third one we set

$$(2.9) \quad x'_i = g_{ij} x'^j; \quad n_i = g_{ij} n^j,$$

and

$$(2.10) \quad c^1 = \begin{vmatrix} x'_2 & x'_3 \\ n_2 & n_3 \end{vmatrix}, \quad c^2 = \begin{vmatrix} x'_3 & x'_1 \\ n_3 & n_1 \end{vmatrix}, \quad c^3 = \begin{vmatrix} x'_1 & x'_2 \\ n_1 & n_2 \end{vmatrix}, \quad c = g_{ij} c^i c^j.$$

At this point we suppose M is an orientable manifold. Then consider two overlapping coordinate systems $(\mathcal{U}; x^i)$ and $(\tilde{\mathcal{U}}; \tilde{x}^i)$ such that the Jacobian determinant function $J(x, \tilde{x}) = \det [\partial \tilde{x}^i / \partial x^j]$ is positive. By straightforward calculations we deduce

$$c^i = \frac{1}{J} \tilde{c}^j \frac{\partial x^i}{\partial \tilde{x}^j} \quad \text{and} \quad c = \frac{1}{J^2} \tilde{c} \quad \text{on } \mathcal{U} \cap \tilde{\mathcal{U}}.$$

This enables us to define the unit Finsler vector field $b = b^i \partial/\partial y^i$ where $b^i = c^i \sqrt{c}$, $i \in \{1, 2, 3\}$. It is easy to check that b is orthogonal to both $\partial/\partial v$ and n .

We call b the *binormal Finsler vector field* along C . Moreover, the bases $\{\partial/\partial v, n, b\}$ and $\{\partial/\partial y^1, \partial/\partial y^2, \partial/\partial y^3\}$ have the same orientation. Thus we are entitled to call $\mathcal{F} = \{\partial/\partial v, n, b\}$ the *Frenet frame* for the curve C in $F^{(3)}$.

Taking into account that ∇ from the Cartan connection is a metric connection and using (2.2) we obtain

$$(2.11) \quad \nabla_{\partial/\partial s} n = -k \frac{\partial}{\partial v} + \tau b,$$

and

$$(2.12) \quad \nabla_{\partial/\partial s} b = -\tau n,$$

where τ is a smooth function given by

$$(2.13) \quad \tau(s) = g(\nabla_{\partial/\partial s} n, b)(s) = g_{ij}(s) b^i(s) \left(\frac{dn^j}{ds} + n^k(s) B_k^j(s) \right).$$

We call τ the *torsion* of C and thus (2.2), (2.11) and (2.12) become the *Frenet formulas* for the curve C in Finsler manifold $F^{(3)}$.

3. Curves in $F^{(3)} = (\mathbb{R}^3, F)$

We consider a curve C in the Finsler manifold $F^{(3)} = (\mathbb{R}^3, F)$ given by equations (2.1). As we have seen in the previous section the vanishing of curvature of C implies that C is a geodesic and viceversa. In the first part of this section we show that the vanishing of torsion of C implies that C lies in a plane of $\mathbb{R}^3$. To this end we first consider a vector subbundle D of rank 2 in $VT\mathbb{R}^3|_C$. Denote by $D^\perp$ the orthogonal complementary vector bundle to D in $VT\mathbb{R}^3|_C$. Then we say that D is a *parallel vector bundle* along C if for any $s \in (-\varepsilon, \varepsilon)$ we have

$$(3.1) \quad g(s)(V, W) = 0,$$

for any V and W from the fibres of D and $D^\perp$ at points $(x(s), vx'(s))$ and $(x(0), x'(0))$ respectively.

Theorem 3.1. *Let C be a curve in $F^{(3)} = (\mathbb{R}^3, F)$ and D be a parallel vector bundle along C such that $\partial/\partial v \in \Gamma(D)$. Then C lies in a plane of $\mathbb{R}^3$.*

Proof. Without loss of generality we may assume that $D(0) = \text{Span}\{\partial/\partial y^1, \partial/\partial y^2\}$ at $(x(0), x'(0))$. As D is parallel along C we deduce that $D(0) = \text{Span}\{\partial/\partial y^1, \partial/\partial y^2\}$ at $(x(s), vx'(s))$. Now suppose there exists a point $x(s) \in C$ such that $x'^3(s) \neq 0$. Then from (1.6) it follows that $\partial/\partial v$ does not lie in $D(s)$, which is a contradiction. Hence at any point of C we have $x'^3(x) = 0$ which means that C lies in a plane of $\mathbb{R}^3$.

Theorem 3.2. *Let C be a curve in $F^{(3)} = (\mathbb{R}^3, F)$ such that the torsion τ is everywhere zero on $(-\varepsilon, \varepsilon)$. Then C lies in some plane of $\mathbb{R}^3$.*

P r o o f As $\tau = 0$ the Frenet equation (2.11) becomes

$$(3.2) \quad \nabla_{\partial/\partial s} n = -k \frac{\partial}{\partial v}.$$

Next denote by D the vector bundle spanned by the pair $\{\partial/\partial v, n\}$ along C . Consider an arbitrary non zero vector $W \in D^\perp(0)$ and denote by the same letter the constant vector field $W(s, v) = W$ at any $(x(s), vx'(s))$. Taking into account that ∇ is a metric connection and using (2.2) and (3.2) we obtain

$$(3.3) \quad \begin{cases} \frac{d}{ds} \left(g \left(\frac{\partial}{\partial v}, W \right) \right) (s) = g \left(\nabla_{\partial/\partial s} \frac{\partial}{\partial v}, W \right) (s) = kg(n, W)(s), \\ \frac{d}{ds} (g(n, W))(s) = g \left(\nabla_{\partial/\partial s} n, W \right) (s) = -kg \left(\frac{\partial}{\partial v}, W \right) (s). \end{cases}$$

Since $g(n, W)(0) = g(\partial/\partial v, W)(0) = 0$, the uniqueness of solutions of the system (3.3) implies $g(n, W)(s) = g(\partial/\partial v, W)(s) = 0$ for any $s \in (-\varepsilon, \varepsilon)$. Hence D is parallel along C . As $\partial/\partial v \in \Gamma(D)$, the assertion follows from Theorem 3.1.

From now on we suppose F depends only on y^i , that is, $F^{(3)} = (\mathbb{R}^3, F)$ is a Minkowski space. Then from (1.13) we deduce $G^i = 0$ and $N_j^i = 0$. Hence (2.4) becomes

$$(3.4) \quad k(s) = \left\{ g_{ij}(s) x''^i(s) x''^j(s) \right\}^{1/2}$$

Next, from (2.6) and (2.8) we infer

$$(3.5) \quad n^i(s) = \frac{1}{k} x''^i(s),$$

and

$$(3.6) \quad B_j^i(s) = C_{jk}^i x''^k(s),$$

respectively. Using (3.5) and (3.6) in (2.13) and taking into account that $g(n, b) = 0$ we obtain

$$(3.7) \quad \tau(s) = \frac{1}{k(s)} g_{ij}(s) b^i(s) \left(x'''^j(s) + C_{jk}^i(s) x''^h(s) x''^k(s) \right).$$

Moreover, by using (2.9), (2.10) and (3.5) we derive

$$c^1 = \frac{\det[g]}{k} \begin{vmatrix} g^{11} & g^{12} & g^{13} \\ x'^1 & x'^2 & x'^3 \\ x''^1 & x''^2 & x''^3 \end{vmatrix}, \quad c^2 = \frac{\det[g]}{k} \begin{vmatrix} g^{21} & g^{22} & g^{23} \\ x'^1 & x'^2 & x'^3 \\ x''^1 & x''^2 & x''^3 \end{vmatrix},$$

$$c^3 = \frac{\det[g]}{k} \begin{vmatrix} g^{31} & g^{32} & g^{33} \\ x'^1 & x'^2 & x'^3 \\ x''^1 & x''^2 & x''^3 \end{vmatrix}.$$

Rmark 3.1. It is easy to check that, in particular, for $F(y) = \{(y^1)^2 + (y^2)^2 + (y^3)^2\}^{1/2}$ we get from (3.4) and (3.7) the well-known formulas for the curvature and torsion of a curve in a three-dimensional Euclidean space.

Finally, we shall find formulas for curvature and torsion of a curve given by local parametric equations (1.1) in a Minkowski space. First, by using (1.2) we deduce

$$(3.8) \quad x'^i = \frac{\dot{x}^i}{F},$$

$$(3.9) \quad x''^i = \frac{1}{F^3} (F\ddot{x}^i - \dot{F}\dot{x}^i),$$

$$(3.10) \quad x'''^i = \frac{1}{F^5} (F^2\ddot{\ddot{x}}^i - 3F\dot{F}\ddot{x}^i + (3(\dot{F})^2 - F\ddot{F})\dot{x}^i).$$

Then using (3.8) – (3.10), (1.19) and (1.23) we obtain that the curvature and torsion of a curve C in a Minkowski space are given by

$$(3.11) \quad k = \frac{1}{F^3} \{g_{ij} (F\ddot{x}^i - \dot{F}\dot{x}^i) (F\ddot{x}^j - \dot{F}\dot{x}^j)\}^{1/2},$$

and

$$(3.12) \quad \tau = \frac{1}{kF^5} g_{ij} b^i \left\{ F^2\ddot{\ddot{x}}^j - 3F\dot{F}\ddot{x}^j + (3(\dot{F})^2 - F\ddot{F})\dot{x}^j + \right. \\ \left. + C_{hk}^j (F\ddot{x}^h - \dot{F}\dot{x}^h) (F\ddot{x}^k - \dot{F}\dot{x}^k) \right\},$$

respectively, where $b^i = c^i \sqrt{c}$ with $c = g_{ij} c^i c^j$ and

$$c^1 = \frac{\det[g]}{kF^3} \begin{vmatrix} g^{11} & g^{12} & g^{13} \\ \dot{x}^1 & \dot{x}^2 & \dot{x}^3 \\ \ddot{x}^1 & \ddot{x}^2 & \ddot{x}^3 \end{vmatrix}, \quad c^2 = \frac{\det[g]}{kF^3} \begin{vmatrix} g^{21} & g^{22} & g^{23} \\ \dot{x}^1 & \dot{x}^2 & \dot{x}^3 \\ \ddot{x}^1 & \ddot{x}^2 & \ddot{x}^3 \end{vmatrix}, \\ c^3 = \frac{\det[g]}{kF^3} \begin{vmatrix} g^{31} & g^{32} & g^{33} \\ \dot{x}^1 & \dot{x}^2 & \dot{x}^3 \\ \ddot{x}^1 & \ddot{x}^2 & \ddot{x}^3 \end{vmatrix}.$$

REFERENCES

1. Bejancu A., *Geometry of Finsler Subspaces (I)*. An. Șt. Univ. "Al. I. Cuza" Iași, **32**, s. Ia, Mat., 69-83 (1986).
2. Bejancu A., *Finsler Geometry and Applications*. Ellis Horwood, New York, 1990.
3. Matsumoto M., *Foundations of Finsler Geometry and Special Finsler Spaces*. Kaiseisha Press, Shigaken, 1986.
4. Rund H., *The Differential Geometry of Finsler Spaces*. Springer, Berlin, 1959.
5. Taylor J. H., *A Generalization of Levi-Civita's Parallelism and the Frenet Formulas*. Trans. Amer. Math. Soc., **27**, 246-264 (1925).

GEOMETRIA CURBELOR ÎN $F^{(3)}$

(Rezumat)

Se construiește reperul Frenet și se obțin ecuațiile Frenet pentru a curbă C într-o varietate Finsler 3-dimensională $F^{(3)}$. Se demonstrează două teoreme de reducere a codimensiunii imersiei și se deduc formule explicite de calcul pentru curbura și torsiunea curbei C .

HARMONIC MAPS BETWEEN ALMOST
HYPERBOLIC HERMITIAN MANIFOLDS

BY

CORNELIA-LIVIA BEJAN and MICHÈLE BENYOUNES

Dedicated to the memory of Prof. Dumitru MANGERON

Abstract. The almost hyperbolic Hermitian manifolds are particular semi-Riemannian manifolds, which appeared for the first time in [4]. We prove in [2] that any para-holomorphic map between two almost hyperbolic Hermitian manifolds satisfying a certain condition is harmonic. The result is applied here to the case of the total space of the cotangent bundle. Some examples are also given.

Key words: harmonic map, hyperbolic Hermitian manifold.

1. Almost Hyperbolic Hermitian Manifolds

Let (M, G) be a semi-Riemannian m -dimensional manifold, that is the metric G is indefinite. If $P : \Gamma(TM) \rightarrow \Gamma(TM)$ is an endomorphism of the $F(M)$ -module $\Gamma(TM)$ of all vector fields of M , such that

$$(1.1) \quad \begin{aligned} P^2 &= I \text{ (Identity)} ; \\ P &\neq \pm I \text{ and } G(PX, Y) + G(X, PY) = 0, \quad \forall X, Y \in \Gamma(TM), \end{aligned}$$

then (M, P, G) is called an *almost hyperbolic Hermitian manifold*. For being hyperbolic Hermitian (resp. hyperbolic Kaehler), the integrability of P (resp. the parallelism of P with respect to the Levi-Civita connection of G) is required. Some examples of almost hyperbolic Hermitian manifolds which are not hyperbolic Kaehler are given in [1]. Any almost hyperbolic Hermitian manifold satisfies: 1) $m = 2r$; 2) the metric G is of signature (r, r) ; 3) the eigendistributions T^+M and T^-M of P , corresponding to its eigenvalues 1 and -1, respectively, are of the same dimension r ; 4) M carries an almost symplectic structure Ω defined by

$$(1.2) \quad \Omega(X, Y) = G(X, PY), \quad \forall X, Y \in \Gamma(TM).$$

These manifolds appeared for the first time in [4] and later on several authors have dealt with their classification, their submanifolds (which can be either degenerate or nondegenerate) and their applications to theoretical physics.

We recall now the notion of harmonic map.

If $F : M \rightarrow N$ is a differentiable map between the semi-Riemannian manifolds (M, G) and (N, H) , then $dF \in \Gamma(T^*M \otimes F^{-1}TN)$, i.e. dF is a 1-form on M with values in the pull-back $F^{-1}TN$. If $(\mathcal{U}, x^1, \dots, x^m)$ and $(\mathcal{V}, u^1, \dots, u^n)$ are two local charts on M and N , respectively, then locally

$$dF = \frac{\partial F^\alpha}{\partial x^i} dx^i \otimes \frac{\partial}{\partial u^\alpha}.$$

The second quadratic form ∇dF is obtained from dF by using the Levi-Civita connection ∇^M on M and the induced connection $\nabla^{F^{-1}TN}$ on $F^{-1}TN$ from the Levi-Civita connection ∇^N on N , [3]:

$$(1.3) \quad (\nabla dF)(X, Y) = \nabla_X^{F^{-1}TN} dF(Y) - dF((\nabla_X^M Y)), \quad \forall X, Y \in \Gamma(TM).$$

∇dF is a symmetric bilinear map from TM to $F^{-1}TN$ whose local expression is

$$(1.4) \quad (\nabla dF)_{ij}^\alpha = \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \frac{\partial F^\alpha}{\partial x^t} \Gamma_{ij}^t + \Gamma_{\beta\gamma}^\alpha \frac{\partial F^\beta}{\partial x^i} \frac{\partial F^\gamma}{\partial x^j},$$

where $(\nabla dF)_{ij}^\alpha \partial/\partial u^\alpha = (\nabla dF)(\partial/\partial x^i, \partial/\partial x^j)$ and Γ_{ij}^t and $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols of G and H , respectively.

The tension τ of F is the trace of the second quadratic form of F and it is expressed locally by

$$\tau^\alpha = G^{ij} (\nabla dF)_{ij}^\alpha.$$

We recall that F is called *harmonic* (resp. *totally geodesic*) if $\tau = 0$ (resp. $\nabla dF = 0$).

As the harmonic maps are solutions of the nonlinear system

$$(1.5) \quad \tau = \text{trace } \nabla dF = 0$$

which is elliptic in the Riemannian case and hyperbolic in the case of indefinit metrics, the later case is less studied than the first one, because of the difficulties arising there. The purpose of the present paper is to give some sufficient conditions for obtaining harmonic maps.

If (M, P, G) and (N, Q, H) are two almost hyperbolic Hermitian manifolds, then any differentiable map $F : M \rightarrow N$ is hyperbolic holomorphic, provided its differential intertwines with P and Q , i.e.

$$(1.6) \quad dF \circ P = Q \circ dF;$$

this condition is equivalent to the preserving of distributions

$$(1.7) \quad dF(T^+M) \subseteq T^+N \quad \text{and} \quad dF(T^-M) \subseteq T^-N.$$

We say that an almost hyperbolic Hermitian manifold (M, P, G) satisfies the condition (C) if

$$(C) \quad (\nabla_X P)(Y) = (\nabla_Y P)X = 0, \quad \forall X \in \Gamma(T^+M), Y \in \Gamma(T^-M).$$

Remark. The manifolds satisfying the condition (C) belong to the classification [1].

Theorem 1 [2]. *If (M, P, G) and (N, Q, H) are two almost hyperbolic Hermitian manifolds satisfying the condition (C), then any hyperbolic holomorphic map between them is harmonic.*

The idea of the proof is to consider the algebra of the para-complex numbers $A = \{a + \varepsilon b/a, b \in \mathbb{R}\}$, with $\varepsilon = \pm 1$. If P^A denotes the A -linear extension of P to $TM^A = TM \otimes_{\mathbb{R}} A$, then $TM^A = T'M \otimes T''M$, where $T'M$ and $T''M$ are the eigendistributions of P^A , corresponding to the eigenvalues $\pm \varepsilon$, respectively. As M is of even dimension $2r$, then any local chart of it has an orthonormal frame of vector fields $\{X_1, \dots, X_r, PX_1, \dots, PX_r\}$. We obtain:

$$(1.8) \quad \tau = \sum_{i=1}^r (\nabla dF)(\bar{Z}_i, Z_i) = \sum_{i=1}^r \nabla_{\bar{Z}_i}^{F^{-1}TN} dF(Z_i) - dF(\nabla_{\bar{Z}_i}^M Z_i),$$

where $Z_i = X_i + \varepsilon PX_i$ and $\bar{Z}_i = X_i - \varepsilon PX_i$, $i = 1, \dots, r$.

By applying an equivalent relation of (1.4) and an equivalent one of (C), we get

$$\tau \in \Gamma(F^{-1}T'N) \cap \Gamma(F^{-1}T''N) = \{0\}$$

and therefore F is harmonic.

Remark. This result is similar to that one obtained in [9], concerning the structure of Kentaro Yano type ($K^3 + K = 0$), compatible with the Riemannian metric of the manifold. Clearly, the para-Hermitian structure P is not a f -structure in sense of Yano.

Now, by taking into account that any hyperbolic Kähler manifold satisfies the condition (C), it follows

Corollary 2. *Any hyperbolic holomorphic map between hyperbolic Kähler manifolds is harmonic.*

Remark. This result corresponds to the well known one, [3], from the Kähler case. There are some other related results in the elliptic case (see [5]) having correspondents in the hyperbolic case, but we proceed now with some examples.

Examples On the torus $T^{1,1} = S^1 \times S^1$ endowed with the indefinite metric $G = d\theta^2 - d\varphi^2$, we define the structure P as an endomorphism of $\Gamma(T(T^{1,1}))$ by $P(a\partial/\partial\theta + b\partial/\partial\varphi) = a\partial/\partial\varphi + b\partial/\partial\theta$ for any real differentiable functions $a, b \in \mathcal{F}(T^{1,1})$. It follows that $(T^{1,1}, P, G)$ provides an example of a hyperbolic Kähler manifold.

Now, we consider the pseudo-Euclidean space $\mathbb{R}^{2,1}$ to be the 3-dimensional real vector space with the inner product $\langle x, y \rangle = x^1 y^1 + x^2 y^2 - x^3 y^3$. Then another example we construct here is the pseudosphere

$$S^{1,1} = \{x \in \mathbb{R}^{2,1} \mid \langle x, x \rangle = 1\} = S^1 \times \mathbb{R},$$

endowed with the indefinit metric $H = \cosh^2 t - dt^2$, on which we define the structure Q as an endomorphism of $\Gamma(T(S^{1,1}))$ by $Q(a \partial/\partial u + b \partial/\partial t) = a \cosh t \partial/\partial t + b/\cosh t \partial/\partial u$, for any differentiable functions $a, b \in \mathcal{F}(S^{1,1})$. We get that $(S^{1,1}, Q, H)$ is a hyperbolic K hler manifold.

In [8], the family of maps $F_{m,n} : (T^{1,1}, G) \rightarrow (S^{1,1}, H)$ defined by

$$(1.9) \quad F_{m,n}(\theta, \varphi) = (m\theta + n\varphi, \alpha(\varphi)), \quad \forall m, n \in \mathbb{Z}$$

is studied and the condition of harmonicity is found:

$$\alpha''(\varphi) + (m^2 + n^2) \cosh \alpha(\varphi) \sinh \alpha(\varphi) = 0.$$

In view of Corollary 2, the hyperbolic holomorphic maps are found only among the harmonic ones. By calculus, we obtain that the set of hyperbolic holomorphic maps selected from the family (1.9) can be characterized by

$$(1.10) \quad n = 0 \quad \text{and} \quad \alpha' = m \cosh \alpha, \quad m \in \mathbb{Z}.$$

As on one side the group $[T^{1,1}, S^{1,1}]$ of all homotopy classes of the maps from $T^{1,1}$ to $S^{1,1}$ is isomorphic with $\mathbb{Z} \times \mathbb{Z}$ and on the other side $\alpha = 0$ is a periodic solution of (1.10) whose associated map $F_{m,0}$ represents the homotopy class $\{m\} \times \{0\} \subset \mathbb{Z} \times \{0\}$, then from Corollary 2 we obtain

Proposition 3. *The group of all homotopy classes $[T^{1,1}, S^{1,1}] = \mathbb{Z} \times \mathbb{Z}$ contains a subgroup, corresponding to $\mathbb{Z} \times \{0\}$, of homotopy classes that can be represented by hyperbolic holomorphic maps.*

Remark. For the family of maps

$$(1.11) \quad f : S^{1,1} \rightarrow S^{1,1} \quad \text{defined by} \quad f(u(t)) = (\phi(u), \alpha(t)),$$

the condition of harmonicity is given in [8], as follows:

$$\alpha''(t) + [p \tanh t] \alpha'(t) - \frac{\lambda}{\cosh^2 t} \sinh \alpha(t) \cosh \alpha(t) = 0.$$

By calculus, we obtain that the hyperbolic holomorphic maps are characterized by the condition

$$\phi = \text{Identity} \quad \text{and} \quad \alpha' = \cosh \alpha.$$

2. Application to the Cotangent Bundle

Let $\pi : T^*\mathcal{M} \rightarrow \mathcal{M}$ be the cotangent bundle of the r -dimensional manifold $\mathcal{M}$ which is endowed with symmetric linear connection ∇ . If $VT^*\mathcal{M} = \text{Ker } d\pi$

denotes the vertical bundle, then ∇ defines the horizontal bundle $HT^*\mathcal{M}$ which gives the splitting $TT^*\mathcal{M} = VT^*\mathcal{M} \oplus HT^*\mathcal{M}$. Let $P : \Gamma(TT^*\mathcal{M}) \rightarrow \Gamma(TT^*\mathcal{M})$ be an endomorphism acting as +Identity and -Identity on $VT^*\mathcal{M}$ and $HT^*\mathcal{M}$, respectively. Assume $i = 1, \dots, r$. Any local chart (U, x^i) of $\mathcal{M}$ induces a local chart $(\pi^{-1}(U), q^i, p_i)$, with $q^i = x^i \circ \pi$ and p_i are the vector space coordinates of $p \in \pi^{-1}(U) \subset T^*\mathcal{M}$ with respect to the natural frame $\{dx^i\}$. The canonical symplectic form of $T^*\mathcal{M}$ is given locally by $dq^i \wedge dp_i$. The horizontal (resp. vertical) lift of the vector field $X = X^i \partial/\partial q_i \in \Gamma(TM)$ (resp. 1-form $\theta = \theta_i dx^i \in \Gamma(T^*\mathcal{M})$) is defined by $X^H = X^i \delta/\delta q^i$ (resp. $\theta^V = \theta_i \partial/\partial p_i$), with $\delta/\delta q^i = \partial/\partial q^i + p_k \Gamma_{ij}^k \partial/\partial p_j$, where Γ_{ij}^k are the Cristoffel symbols of ∇ [10]. Therefore $\{\partial/\partial p_i = (dx^i)^V\}_i$ and $\{\delta/\delta q_i = (\partial/\partial x^i)^H\}_i$ are local frames of $VT^*\mathcal{M}$ and $HT^*\mathcal{M}$, respectively. The Riemann extension of ∇ [7] is an indefinite metric on the total space of the cotangent bundle $T^*\mathcal{M}$, defined by

$$(2.1) \quad G(X^H, Y^H) = G(\theta^V, \sigma^V) = 0, \quad G(X^H, \theta^V) = G(\theta^V, X^H) = \theta(X), \\ \forall X, Y \in \Gamma(TM), \quad \theta, \sigma \in \Gamma(T^*\mathcal{M}).$$

The Levi-Civita connection D of G is given by

$$(2.2) \quad D_{\theta^V} \sigma^V = D_{\theta^V} X^H = 0, \quad D_{X^H} \theta^V = (\nabla_X \theta)^V; \\ D_{X^H} Y^H = (\nabla_X Y)^H + A(X^H, Y^H),$$

where A is a bilinear map from $HT^*\mathcal{M}$ to $VT^*\mathcal{M}$, having the local expression

$$(2.3) \quad A\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right) = -p_l R_{ijk}^l \frac{\partial}{\partial p_k},$$

with R_{ijk}^l denoting the local coordinates components of the curvature tensor field R of ∇ [6]. Therefore we get

Proposition 4. *If $\mathcal{M}$ is a manifold endowed with a symmetric linear connection ∇ , then, under above notations, $(T^*\mathcal{M}, P, G)$ is an almost para-Hermitian manifold (see [1]) satisfying the condition (C). (Its associated symplectic form ω coincides, up to the sign, to the canonical symplectic form.)*

As a consequence, we get

Theorem 5. *If the manifolds $\mathcal{M}$ and $\mathcal{N}$ are endowed with symmetric linear connections and the total spaces of their cotangent bundles $T^*\mathcal{M}$ and $T^*\mathcal{N}$ carry the induced indefinite metrics defined by the corresponding Riemann extensions, then any map $F : T^*\mathcal{M} \rightarrow T^*\mathcal{N}$ is harmonic, provided its differential dF preserves both the vertical and the horizontal distribution, i.e.*

$$(2.4) \quad dF(VT^*\mathcal{M}) \subseteq VT^*\mathcal{N} \quad \text{and} \quad dF(HT^*\mathcal{M}) \subseteq HT^*\mathcal{N}.$$

The proof comes out from the construction of the structures P and Q on $T^*\mathcal{M}$ and $T^*\mathcal{N}$, respectively, like in Proposition 4 and from the fact that the equalities $T^+T^*\mathcal{M} = VT^*\mathcal{M}$ and $T^-T^*\mathcal{M} = HT^*\mathcal{M}$ (similarly for $\mathcal{N}$) allow us to apply Corollary 2.

By taking the total space of the tangent bundles of two Riemannian manifold, instead of the cotangent ones, a corresponding result of Theorem 5 is obtained in the forthcoming paper [2].

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REFERENCES

1. Bejan C. L., *Some Examples of Manifolds with Hyperbolic Structures*. Rendiconti Matematica Roma, **VII**, 14, 557-565 (1994).
2. Bejan C. L., Benyounes M., *Harmonic Maps Between Almost Para-Hermitian Manifolds* Kluwer Acad. Publ., Proc. Conf. Diff. Geom., Budapest, 1996 (1998), to appear.
3. Eells J., Lemaire L., *Selected Topics in Harmonic Maps*. Conf. Board Math. Sc., AMS, **50** (1983).
4. Libermann P., *Sur le problème d'équivalence de certains structures infinitésimales*. Ann. Mat., **36** (1954).
5. Lichnerowicz A., *Choix d'œuvres mathématiques*. Hermann, Edit. Sci. Arts, 1982.
6. Oproiu V., *On the Harmonic Sections of Cotangent b Bundles*. M. Dekker, New York, 1973.
7. Patterson E. M., Walker A. G., *Riemann Extension*. Quart. J. Math. Oxford, **2**, **3**, 19-28 (1952).
8. Ratto A., *Equivariant Harmonic Maps Between Manifolds with Metrics of (p, q) -Signature*. Ann. Inst. H. Poincaré, **6**, **6**, 503-524 (1989).
9. Rawnsley J., *f-structures, f-twistor Spaces and Harmonic Maps*. Lect. Notes in Math., Springer Verlag 1985.
10. Yano K., Ishihara S., *Tangent and Cotangent Bundles*. M. Dekker, New York, 1973.

APLICAȚII ARMONICE ÎNTRE VARIETĂȚI APROAPE HERMITIENE HIPERBOLICE

(Rezumat)

Varietăți aproape hermitiene hiperbolice sunt cazuri particulare de varietăți semi-Riemanniene care apar pentru prima dată în [4]. În [2] se demonstrează că orice aplicație paraholomorfă între două varietăți aproape hermitiene hiperbolice satisfăcând o anumită condiție, este armonică. Acest rezultat se aplică aici la cazul spațiului total al fibratului cotangent la o varietate diferențiabilă. Sunt prezentate două exemple importante.

QUASI-LINEAR FUNCTIONS

BY

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Abstract. This note proves some properties of quasi-linear spaces and quasi-linear functions between such spaces.

Key words: quasi-linear space, quasi-linear function.

Let $(K, +, \cdot)$ be a field with distinct 0 and 1. Let $(X, +)$ be a commutative additive group with θ_X or θ the zero of addition and $(-x) \in X$ the opposite of $x \in X$. We call $(X, +, \cdot, K)$ a *pre-linear space over K* if: $K \times X \rightarrow X$ is a scalar multiplication of elements of X satisfying:

$$\begin{aligned} \forall(\alpha, x, y) \in K \times X^2, & \quad \alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y, \\ \forall(\alpha, \beta, x) \in K^2 \times X, & \quad \alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x, \\ \forall x \in X, & \quad 1 \cdot x = x. \end{aligned}$$

If, in addition, the scalar multiplication satisfies

$$(1) \quad \forall(\alpha, \beta, x) \in K^2 \times X, \quad (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x,$$

respectively

$$(2) \quad \forall(\alpha, \beta, \gamma, x) \in K^3 \times X, \quad (\alpha + \beta + \gamma) \cdot x = \alpha \cdot x + \beta \cdot x + \gamma \cdot x,$$

then the pre-linear space X is called a *linear space over K* , respectively, a *quasi-linear space over K* [1], [2].

Example 1. Every commutative periodic group of order 2 is a quasi-linear space over an arbitrary field [1]. Indeed, if $(G, \circ)$ is a periodic group of order 2 and K is a field, then $(G, +, \cdot, K)$ is a quasi-linear space over K with respect to the laws $x + y = x \circ y$ and $\alpha \cdot x = x$ for each $(x, y) \in G^2$ and $(\alpha, x) \in K \times G$. The scalar multiplication satisfies (2), but not (1), therefore G is not a linear space.

Example 2. If $(G, \circ)$ is a commutative periodic group of order 2 with neutral element e , $G \neq \{e\}$ and $(X, +, \cdot, K)$ is a linear space, then $(G \times X, +, \cdot, K)$ is a quasi-linear (but not linear) space over K , with respect to the laws

$$\begin{aligned}(x, u) + (y, v) &= (x \circ y, u + v), \\ \alpha \cdot (x, u) &= (x, \alpha \cdot u),\end{aligned}$$

for every $(x, u), (y, v)$ in $G \times X$ and $\alpha \in K$. In this case, we have $0 \cdot (x, u) = (x, \theta)$, where θ is the zero in X , therefore the scalar multiplication with the scalar 0 of the element (x, u) is not the neutral element (e, θ) . As a special case, if $G = Z_2 = \{\hat{0}, \hat{1}\}$ is the periodic group of order 2 of residues classes modulo 2, then $(Z_2 \times \mathbb{R}, +, \cdot, \mathbb{R})$ is a quasi-linear (but not linear) space. The neutral element of addition in $Z_2 \times \mathbb{R}$ is $(\hat{0}, 0)$ and the equation $0 \cdot (\hat{a}, x) = (\hat{0}, 0)$ has an infinity of solutions $(\hat{0}, x)$ for each $x \in \mathbb{R}$, and the element $0 \cdot (\hat{a}, x)$ is distinct of $(\hat{0}, 0)$ for $\hat{a} = \hat{1}$.

Accordingly, if x is an element of a quasi-linear space X , the element $0 \cdot x$ of X is not necessarily equal to θ , as it happens in linear spaces.

Therefore, it is natural to consider the set

$$X^0 = \{0 \cdot x; x \in X\},$$

playing the role of θ in the linear spaces [2].

It is obvious that for each finite system $(x_i)_{i \in I}$ of elements of a pre-linear space over a field K , the element $\sum_{i \in I} x_i$ is well defined in X and for every $\alpha \in K$ we have $\alpha \cdot \sum_{i \in I} x_i = \sum_{i \in I} (\alpha \cdot x_i)$.

The proofs of the following statements are immediate.

Proposition 1. *Let $(X, +, \cdot, K)$ be a quasi-linear space. Then:*

- (i) $\forall x \in X, \theta = 0 \cdot x + 0 \cdot x \in X^0$;
- (ii) $\forall (\alpha, x) \in K \times X^0, \alpha \cdot x = x$;
- (iii) $x \in X, \alpha \cdot x \in X^0, x \notin X^0$ imply $\alpha = 0$;
- (iv) $x \in X, \alpha \cdot x = \theta, x \neq \theta$ imply $\alpha = 0$;
- (v) $x \in X, \alpha \cdot x \in X^0, \alpha \neq 0$ imply $x \in X^0$;
- (vi) $x \in X, \alpha \cdot x = \theta, \alpha \neq 0$ imply $x = \theta$;
- (vii) $\forall (\alpha, \beta, x) \in K^2 \times X, (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x + 0 \cdot x$;
- (viii) $\forall (\alpha, x) \in K \times X, \alpha(-x) = -(\alpha \cdot x), (-\alpha) \cdot x = -(\alpha \cdot x)$;
- (ix) if $(\alpha_i)_{1 \leq i \leq m}$ is a system of elements of K and $x \in X$, then the element $\left(\sum_{i=1}^m \alpha_i\right) \cdot x$ is $\left(\sum_{i=1}^m \alpha_i \cdot x\right)$ for m odd and $\left(\sum_{i=1}^m \alpha_i \cdot x\right) + 0 \cdot x$ for m even.

Definition 1. Let $(X, +, \cdot, K)$ a pre-linear space, U a subset of X , $\oplus : U \times U \rightarrow X$ and $\odot : K \times U \rightarrow X$ be the restrictions to $U \times U$, respectively $K \times U$, of the laws $+$: $X \times X \rightarrow X$, $\cdot$: $K \times X \rightarrow X$. The subset U is called a *subspace* of the space X if $\oplus$ and $\odot$ have the values in U or $(U, +, \cdot, K)$ is a pre-linear space over K .

As a consequence, the following statement holds:

Proposition 2. *If $(X, +, \cdot, K)$ is a pre-linear space and $U \subset X$, the following statements are equivalent:*

- (a) U is a subspace of X ;
- (b) for each $(x, y) \in U^2$ and $(\alpha, x) \in K \times U$ we have $x + y \in U$, $\alpha \cdot x \in U$;
- (c) for each $(\alpha, \beta, x, y) \in K^2 \times U^2$ we have $\alpha \cdot x + \beta \cdot y \in U$.

Accordingly, every subspace of X contains θ .

Definition 2. Let $(X, +, \cdot, K)$ a quasi-linear space and U be a subspace of X . The subspace U is called a *linear subspace* if it satisfies (1), respectively, a *quasi-linear subspace* if it satisfies (2), but not (1).

Proposition 3. *Let $(X, +, \cdot, K)$ be a quasi-linear space. Then:*

- (i) if U is a subspace of X , $(x_i)_{1 \leq i \leq n}$ a system of elements of U and $(\alpha_i)_{1 \leq i \leq n}$ a system of scalars of K , we have $\sum_{i=1}^n \alpha_i \cdot x_i \in U$;
- (ii) if $(U_i)_{i \in I}$ is a system of subspaces of X , the subset $\bigcap_{i \in I} U_i$ of X is a subspace of X ;
- (iii) $\{\theta\}$ is a linear subspace of X and X is a quasi-linear subspace of X ;
- (iv) X^0 is a quasi-linear subspace of X ;
- (v) $X_0 = \{x + 0 \cdot x; x \in X\}$ is a linear subspace of X ;
- (vi) $X^0 \cap X_0 = \{\theta\}$.

Definition 3. Let $(X, +, \cdot, K)$ and $(Y, +, \cdot, K)$ quasi-linear spaces and Y^0 be the subspace $Y^0 = \{0 \cdot u; u \in Y\}$. A function $f: X \rightarrow Y$ is called a *quasi-linear function* from X to Y if:

- (i) f is quasi-additive

$$\forall (x, y) \in X^2, \quad f(x + y) - f(x) - f(y) \in Y^0$$

and

- (ii) f is quasi-homogeneous

$$\forall (\alpha, x) \in K \times X, \quad f(\alpha \cdot x) - \alpha \cdot f(x) \in Y^0$$

or, equivalently:

- (i) $\forall (x, y) \in X^2, \exists u \in Y, f(x + y) = f(x) + f(y) + 0 \cdot u$;
- (ii) $\forall (\alpha, x) \in K \times X, \exists u \in Y, f(\alpha \cdot x) = \alpha \cdot f(x) + 0 \cdot u$.

A quasi-linear function f is called a *linear function* if $f(x + y) - f(x) - f(y) = \theta$, $f(\alpha \cdot x) - \alpha \cdot f(x) = \theta$ for every $(x, y) \in X^2$ and $\alpha \in K$.

Proposition 4. *If $(X, +, \cdot, K)$ and $(Y, +, \cdot, K)$ are quasi-linear spaces, $X^0 = \{0 \cdot x; x \in X\}$, $Y^0 = \{0 \cdot u; u \in Y\}$ and $f: X \rightarrow Y$ is a quasi-linear function, then $f(X^0) \subset Y^0$ and $X^0 \subset f^{-1}(Y^0)$.*

Let us observe that if θ_X and θ_Y are the zeros of the additions in X and Y , then $f(\theta_X) \in Y^0$ and $f(\theta_X)$ is, generally, distinct of θ_Y .

Proposition 5. Let $(X, +, \cdot, K)$ and $(Y, +, \cdot, K)$ quasi-linear spaces and $f: X \rightarrow Y$ be a function. Then:

(i) f is quasi-linear if and only if for each $(\alpha, \beta, x, y) \in K^2 \times X^2$ we have $f(\alpha \cdot x + \beta \cdot y) - \alpha \cdot f(x) - \beta \cdot f(y) \in Y^0$;

(ii) if f is quasi-linear, $(x_i)_{1 \leq i \leq n}$ and $(\alpha_i)_{1 \leq i \leq n}$ are systems of elements of X , respectively K , we have

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) - \sum_{i=1}^n \alpha_i \cdot f(x_i) \in Y^0.$$

Definition 4. Let $(X, +, \cdot, K)$ a pre-linear space and A, B be subsets of X . The subset $A + B = \{x + y; x \in A, y \in B\}$ is called the *sum of subsets A and B*.

It is obvious that if $\theta \in B$ then $A \subset A + B$ and if $\theta \in A$ then $B \subset A + B$. Also, if A and B are subspaces then $A + B$ is a subspace.

Proposition 6. Let $(X, +, \cdot, K)$ and $(Y, +, \cdot, K)$ quasi-linear spaces, U a subspace of X , V a subspace of Y and $f: X \rightarrow Y$ be a quasilinear function. Then $f(U) + Y^0$ is a subspace of Y and $f^{-1}(V + Y^0)$ is a subspace of X .

Proof. For each α, β in K and u, v in $f(U) + Y^0$ there exist x, y in U and r, s in Y such that $u = f(x) + 0 \cdot r, v = f(y) + 0 \cdot s, \alpha \cdot u + \beta \cdot v = \alpha \cdot f(x) + \beta \cdot f(y) + 0 \cdot (r + s)$. According to the quasi-linearity of f , there exists $t \in Y$ such that $\alpha \cdot f(x) + \beta \cdot f(y) = f(\alpha \cdot x + \beta \cdot y) + 0 \cdot t$. Hence $\alpha \cdot u + \beta \cdot v = f(\alpha \cdot x + \beta \cdot y) + 0 \cdot (r + s + t) \in f(U) + Y^0$ and because U is a subspace of X , we have $\alpha \cdot x + \beta \cdot y \in U$. Now, for every α, β in K and x, y in $f^{-1}(V + Y^0)$, we have $f(x), f(y)$ in $V + Y^0$ and $\alpha \cdot f(x) + \beta \cdot f(y)$ in $V + Y^0$ because $V + Y^0$ is a subspace of Y . According to the quasi-linearity of f we obtain $f(\alpha \cdot x + \beta \cdot y) \in V + Y^0$ and, hence, $\alpha \cdot x + \beta \cdot y \in f^{-1}(V + Y^0)$. Q.E.D.

Proposition 7. Let $(X, +, \cdot, K)$ and $(Y, +, \cdot, K)$ quasi-linear spaces, $\lambda \in K, f: X \rightarrow Y$ and $g: X \rightarrow Y$ be quasi-linear functions. Then $f + g$ and $\lambda \cdot f$ are quasi-linear functions.

Proof. For each $(\alpha, \beta, x, y) \in K^2 \times X^2$ we have $(f + g)(\alpha \cdot x + \beta \cdot y) = f(\alpha \cdot x + \beta \cdot y) + g(\alpha \cdot x + \beta \cdot y)$ and $(\lambda \cdot f)(\alpha \cdot x + \beta \cdot y) = \lambda \cdot f(\alpha \cdot x + \beta \cdot y)$. According to the quasi-linearity of f and g there exists $(u, v) \in Y^2$ such that $f(\alpha \cdot x + \beta \cdot y) = \alpha \cdot f(x) + \beta \cdot f(y) + 0 \cdot u, g(\alpha \cdot x + \beta \cdot y) = \alpha \cdot g(x) + \beta \cdot g(y) + 0 \cdot v$. Then $(f + g)(\alpha \cdot x + \beta \cdot y) = \alpha \cdot f(x) + \beta \cdot f(y) + \alpha \cdot g(x) + \beta \cdot g(y) + 0 \cdot (u + v) = \alpha \cdot (f + g)(x) + \beta \cdot (f + g)(y) + 0 \cdot (u + v)$ and so $f + g$ is quasi-linear. Also, $(\lambda \cdot f)(\alpha \cdot x + \beta \cdot y) = (\lambda \cdot \alpha) \cdot f(x) + (\lambda \cdot \beta) \cdot f(y) + 0 \cdot u = \alpha \cdot (\lambda \cdot f)(x) + \beta \cdot (\lambda \cdot f)(y) + 0 \cdot u$ that is $\lambda \cdot f$ is quasi-linear. Q.E.D.

Proposition 8. Let $(X, +, \cdot, K), (Y, +, \cdot, K), (Z, +, \cdot, K)$ quasi-linear spaces, $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be quasi-linear functions. Then $g \circ f: X \rightarrow Z$ is a quasi-linear function.

Proof. For each $(\alpha, \beta, x, y) \in K^2 \times X^2$ we have $(g \circ f)(\alpha \cdot x + \beta \cdot y) = g(f(\alpha \cdot x + \beta \cdot y))$. Because of the quasi-linearity of f , there exists $u \in Y$ such that $f(\alpha \cdot x + \beta \cdot y) = \alpha \cdot f(x) + \beta \cdot f(y) + 0 \cdot u$ and then, $(g \circ f)(\alpha \cdot x + \beta \cdot y) = g(\alpha \cdot f(x) + \beta \cdot f(y) + 0 \cdot u)$. According to the quasi-linearity of g there exists $v \in Z$

such that $g(\alpha \cdot f(x) + \beta \cdot f(y) + 0 \cdot u) = \alpha \cdot (g \circ f)(x) + \beta \cdot (g \circ f)(y) + 0 \cdot (g(u) + v)$ that is $g \circ f$ is a quasi-linear function. Q.E.D.

Proposition 9. *Let $(X, +, \cdot, K)$ and $(Y, +, \cdot, K)$ quasi-linear spaces and $f : X \rightarrow Y$ be a quasi-linear bijection. Then the following statements are equivalent:*

- (a) $f^{-1} : X \rightarrow Y$ is quasi-linear;
- (b) $\forall x \in X, \forall u \in Y, \exists x' \in X, f(x) + 0 \cdot u = f(x + 0 \cdot x')$.

Proof. (b) $\Rightarrow$ (a). Because f is bijection, then for each $(u, v) \in Y^2$ there exists $(x, y) \in X^2$ such that $u = f(x)$ or $x = f^{-1}(u)$ and $v = f(y)$ or $y = f^{-1}(v)$. According to the quasi-linearity of f there exists $w \in Y$ such that for each $(\alpha, \beta) \in K^2$ we have $f^{-1}(\alpha \cdot u + \beta \cdot v) = f^{-1}(\alpha \cdot f(x) + \beta \cdot f(y)) = f^{-1}(f(\alpha \cdot x + \beta \cdot y) + 0 \cdot w)$. Using (b), there exists $z \in X$ such that $f(\alpha \cdot x + \beta \cdot y) + 0 \cdot w = f(\alpha \cdot x + \beta \cdot y + 0 \cdot z)$ and $f^{-1}(\alpha \cdot u + \beta \cdot v) = f^{-1}(f(\alpha \cdot x + \beta \cdot y + 0 \cdot z)) = \alpha \cdot x + \beta \cdot y + 0 \cdot z = \alpha \cdot f^{-1}(u) + \beta \cdot f^{-1}(v) + 0 \cdot z$, that is f^{-1} is a quasi-linear function.

(a) $\Rightarrow$ (b). According to (a), for each $(x, u) \in X \times Y$ there exists $y \in X$ such that $f^{-1}(f(x) + 0 \cdot u) = f^{-1}(f(x)) + 0 \cdot f^{-1}(u) + 0 \cdot y = x + 0 \cdot (f^{-1}(u) + y)$. It follows that there exists $x' = f^{-1}(u) + y \in X$ such that $f^{-1}(f(x) + 0 \cdot u) = x + 0 \cdot x'$ and (b) occur. Q.E.D.

Example 3. If $(X, +, \cdot, K)$ is a quasi-linear space and a is a fixed element of X , then the function $f(x) = 0 \cdot a, x \in X$, is non-injective, quasi-linear and it does not satisfy the statement (b) in Proposition 9.

Example 4. If $(X, +, \cdot, K)$ is a quasi-linear space and a is a fixed element of X , then the function $f(x) = x + 0 \cdot a, x \in X$, is bijective, quasi-linear and satisfies the statement (b) in the Proposition 9.

Example 5. If $(X, +, \cdot, K)$ is a quasi-linear space and a is a fixed element of X , then the function $f(x) = 0 \cdot (x + a), x \in X$, is non-injective, quasi-linear and satisfies the statement (b) in Proposition 9.

Example 6. The function $f : Z_2 \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \times \mathbb{R}$ defined by $f(\hat{0}, t) = (A_0, t), f(\hat{1}, t) = (A_1, t), t \in \mathbb{R}$, where $A_1 = \{2k; k \in \mathbb{Z}\}$ and $A_0 = \{2k+1; k \in \mathbb{Z}\}$, is non-injective, quasi-linear and does not satisfy the statement (b) in Proposition 9.

Example 7. The function $f : Z_2 \times \mathbb{R} \rightarrow Z_2 \times \mathbb{R}$ defined by $f(\hat{x}, t) = (\hat{x} + \hat{1}, t), (\hat{x}, t) \in Z_2 \times \mathbb{R}$, is bijective, quasi-linear and satisfies the statement (b) in Proposition 9.

Example 8. The function $f : \{\phi, \mathbb{R}\} \times \mathbb{R} \rightarrow Z_2 \times \mathbb{R}$ defined by $f(A, x) = (\chi_A(x), x), (A, x) \in \{\phi, \mathbb{R}\} \times \mathbb{R}$, is a linear bijection satisfying the statement (b) in Proposition 9, where χ_A is the characteristic function of the set A in $\mathbb{R}$ and $\chi_A(x) \in Z_2$.

REFERENCES

1. Borș D. M., *O generalizare a spațiilor vectoriale*. Bul. Inst. Polit. Iași, **XV (XIX)**, 3-4, s. I (1969).
2. Borș D. M., *Quasivector Spaces over Fields*. Bul. Inst. Polit. Iași, **XVII (XXI)**, 1-2, s. I (1971).

FUNCTII QUASILINIARE

(Rezumat)

Sunt puse în evidență câteva proprietăți ale funcțiilor cuasiliniare între spații cuasiliniare.

A DIRECT METHOD FOR A JORDAN BASIS IN AN n -DIMENSIONAL VECTOR SPACE

BY

I. ONCIULESCU

Abstract. The paper presents a direct method for the determination of a Jordan basis of the form (2) in an n -dimensional vector space over a commutative field.

Key words: Jordan basis, vector space, linear operator, series of eigenvectors.

1. Introduction

We shall denote by X_n an n -dimensional vector space over a commutative field K .

Let $f : X_n \rightarrow X_n$ be a linear operator. If $\lambda_0 \in K$ is an eigenvalue of f , then a vector-system

$$(1) \quad \{a_1, \dots, a_k\}$$

of X_n , satisfying

$$(1') \quad f(a_1) = \lambda_0 a_1, \quad f(a_2) = \lambda_0 a_2 + a_1, \quad f(a_k) = \lambda_0 a_k + a_{k-1}, \quad a_1 \neq 0$$

is called a series of eigenvectors and associated vectors.

A Jordan basis in X_n , corresponding to the operator f , is a basis in X_n of the form

$$(2) \quad \{a_1^1, \dots, a_{n_1}^1, \dots, a_1^p, \dots, a_{n_p}^p\}, \quad n_1 + \dots + n_p = n,$$

where $\{a_1^i, \dots, a_{n_i}^i\}$, $i = \overline{1, p}$, is a series of eigenvectors and associated vectors, corresponding to the eigenvalue λ_i of the operator f .

Suppose that there is a Jordan basis in X_n , corresponding to f , of the form (2). Consider the sets

$$(3) \quad S_\lambda(f) = \{v/v \in X_n, f(v) = \lambda v\}$$

and

$$S_\lambda^1(f) = S_\lambda(f),$$

$$S_\lambda^2(f) = \{v/v \in S_\lambda(f), (\exists)v_1 \in X_n \text{ so that } f(v_1) = \lambda v_1 + v\},$$

$$(4) \quad S_\lambda^k(f) = \{v/v \in S_\lambda(f), (\exists)v_1, \dots, v_{k-1} \in X_n \text{ so that}$$

$$f(v_1) = \lambda v_1 + v, \quad f(v_2) = \lambda v_2 + v_1, \dots, f(v_{k-1}) = \lambda v_{k-1} + v_{k-2}\}, \quad k \geq 3,$$

where λ is in K .

It is immediate the fact that $S_\lambda^k(f)$ is a vector subspace of the vector space $S_\lambda(f)$. We shall determine the dimension of space $S_{\lambda_i}^{n_i}(f)$ (λ_i, n_i are from (2)) and then, using this fact and other properties concerning the series of vectors and associated vectors [5], [6] we obtain the background for a direct method for the determination of a Jordan basis (2).

2. The Dimension of the Spaces $S_\lambda^k(f)$ Involved in the Determination of a Jordan Basis

Consider the linear operator $f: X_n \rightarrow X_n$ and the corresponding spaces (3) and (4).

L e m m a. Suppose that the vector-system

$$(5) \quad \{e_1, e_2, \dots, e_n\}$$

is a basis in X_n and let be

$$(6) \quad a = \sum_{i=1}^n \xi_i e_i, \quad a \in S_\lambda(f).$$

If A is the matrix of the operator f with respect to basis (5) and I_n is the unit n -order matrix, then the vector (6) is in space $S_\lambda^k(f)$ if and only if the linear algebraic system

$$(7) \quad (A - \lambda I_n)^{k-1} [x_1, \dots, x_n]^T = [\xi_1, \dots, \xi_n]^T$$

is a consistent one in K .

P r o o f. We first suppose that the vector (6) is in $S_\lambda^k(f)$. Then there are $k-1$ vectors

$$(8) \quad v_i = \sum_{p=1}^n v_p^i e_p, \quad i = \overline{1, k-1},$$

for which we have

$$(9) \quad f(v_1) = \lambda v_1 + a, \quad f(v_i) = \lambda v_i + v_{i-1}, \quad i = \overline{2, k-1},$$

that is

$$(9') \quad (A - \lambda I_n)[v_1^1, \dots, v_n^1]^T = [\xi_1, \dots, \xi_n]^T,$$

$$(A - \lambda I_n)[v_1^i, \dots, v_n^i]^T = [v_1^{i-1}, \dots, v_n^{i-1}]^T, \quad i = \overline{2, k-1}.$$

From (9') it follows that $[v_1^{k-1}, \dots, v_n^{k-1}]^T$ is a solution of system (7).

Conversely, assume that the system (7) (corresponding to vector (6)) is a consistent one, and let $[x_1, \dots, x_n]^T$ be a solution of its. Then, if we consider the vectors (8) in which $[v_1^i, \dots, v_n^i]^T = (A - \lambda I_n)^{k-1-i}[x_1, \dots, x_n]^T$, $i = \overline{1, k-1}$, these vectors satisfy (9), and consequently vector (6) is in $S_\lambda^k(f)$.

In that follows we assume that, for the linear operator $f: X_n \rightarrow X_n$, there exists in X_n a Jordan basis (2), with the properties

$$(10) \quad \lambda_1 = \lambda_2 = \dots = \lambda_k = \alpha, \quad \alpha \neq \lambda_{k+1}, \dots, \lambda_p$$

and

$$(11) \quad n_1 \leq n_2 \leq \dots \geq n_{i-1} < n_i \leq n_{i+1} \leq \dots \leq n_k.$$

Remark 1. The equalities (10) mean that the number of series of eigenvectors and associated vectors corresponding to the eigenvalue α is k . It is known that

$$(12) \quad k = \dim S_\alpha(f)$$

and the vector-system $\{a_1^1, \dots, a_1^k\}$ is a basis in the space $S_\alpha(f)$ [6], [7].

Theorem 1. Let $f: X_n \rightarrow X_n$ be a linear operator for which there exists a Jordan basis in X_n . Suppose that this basis is of the form (2) and it satisfies conditions (10) and (11). Then we have

$$(12') \quad \dim S_\alpha^{n_i}(f) = k - i + 1.$$

Proof. Obviously, the vector-system

$$(13) \quad \{a_1^i, \dots, a_1^k\}$$

is contained in $S_\alpha^{n_i}(f)$ and it is linearly independent. It remains to prove that every vector $v \in S_\alpha^{n_i}(f)$ can be written as a linear combination of $a_1^i, \dots, a_1^k$.

In view of condition (11), the matrix of f , with respect to the Jordan basis (2), is

$$(14) \quad J = \text{diag} [J_{n_1}(\alpha), \dots, J_{n_{i-1}}(\alpha), J_{n_i}(\alpha), \dots, J_{n_k}(\alpha), J_{n_{k+1}}(\lambda_{k+1}), \dots, J_{n_p}(\lambda_p)],$$

where

$$(14') \quad J_1(\lambda) = [\lambda], \quad J_m(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}, \quad m > 1, \quad \lambda \in K$$

($J_m(\lambda)$ is an m -order Jordan block). It results from (14) that

$J - \alpha I_n = \text{diag} [J_{n_1}(0), \dots, J_{n_{i-1}}(0), J_{n_i}(0), \dots, J_{n_k}(0), J_{n_{k+1}}(\lambda_{k+1} - \alpha), \dots, J_{n_p}(\lambda_p - \alpha)]$,
and then

$$(15) \quad (J - \alpha I_n)^{n_i-1} = \text{diag} [0_{n_1}, \dots, 0_{n_{i-1}}, J_{n_i}^{n_i-1}(0), \dots, J_{n_k}^{n_i-1}(0), \\ J_{n_{k+1}}^{n_i-1}(\lambda_{k+1} - \alpha), \dots, J_{n_p}^{n_i-1}(\lambda_p - \alpha)]$$

(0_m denotes the m -order zero matrix).

Now, let us take a vector $v \in S_\alpha^{n_i}(f)$. Then $v \in S_\alpha(f)$ and consequently it can be written in the form

$$(16) \quad v = \beta_1^i a_1^1 + \dots + \beta_1^{i-1} a_1^{i-1} + \beta_1^i a_1^i + \dots + \beta_1^k a_1^k.$$

Taking into account the Lemma, there exists a system of scalars of K , $(x_1^1, \dots, x_{n_1}^1, \dots, x_1^p, \dots, x_{n_p}^p)$, such that

$$(17) \quad (J - \alpha I_n)^{n_i-1} [x_1^1, \dots, x_{n_1}^1, \dots, x_1^p, \dots, x_{n_p}^p]^T = \\ = [\underbrace{\beta_1^1, 0, \dots, 0}_{n_1 \text{ terms}}, \underbrace{\beta_1^2, 0, \dots, 0, \dots}_{n_2 \text{ terms}}, \dots, \underbrace{\beta_1^{i-1}, 0, \dots, 0}_{n_{i-1} \text{ terms}}, \underbrace{\beta_1^i, 0, \dots, 0}_{n_i \text{ terms}}, \dots, \underbrace{\beta_1^k, 0, \dots, 0}_{n_k \text{ terms}}, \dots, \underbrace{0, \dots, 0}_{n_p \text{ terms}}]^T.$$

It results from (17) that $\beta_1^1 = 0, \dots, \beta_1^{i-1} = 0$ and consequently we have $v = \beta_1^i a_1^i + \dots + \beta_1^k a_1^k$.

3. The Replacement of Some Series of Eigenvectors and Associated Vectors of a Jordan Basis by Other Series of Eigenvectors and Associated Vectors

Suppose that $f : X_n \rightarrow X_n$ is a linear operator for which there exists a Jordan basis in X_n , of the form (2) and also assume the conditions (10) and

$$(18) \quad n_1 = n_2 = \dots = n_{h_1} = m_1 < n_{h_1+1} = \dots = n_{h_1+h_2} = m_2 < \dots < \\ < n_{h_1+h_2+\dots+h_{r-1}+1} = \dots = n_{h_1+h_2+\dots+h_r} = m_r, \quad h_1 + h_2 + \dots + h_r = k,$$

to be satisfied.

Consider in X_n a basis of the form (5), then, the matrix A of the operator f with respect to this basis, and after that we express the spaces $S_\alpha^{m_i}(f)$, $i = \overline{1, r}$, taking into account the Lemma, in the form

$$(19) \quad S_\alpha^{m_i}(f) = \left\{ a = \sum_{i=1}^n \xi_i e_i : a \in S_\alpha(f) \text{ the algebraic system} \right.$$

$$(A - \alpha I_n)^{m_i-1} [x_1, \dots, x_n] = [\xi_1, \dots, \xi_n]^T \text{ is consistent in } K \Big\}.$$

As a consequence of Theorem 1, we have

$$(20) \quad \begin{aligned} \dim S_{\alpha}^{m_1}(f) &= k, \\ \dim S_{\alpha}^{m_i}(f) &= q_i, \quad q_i = k - (h_1 + h_2 + \dots + h_{i-1}), \quad i = \overline{2, r}. \end{aligned}$$

Remark 2. From (20) we have $\dim S_{\alpha}^{m_1}(f) = k$, such that we have

$$(21) \quad S_{\alpha}^{m_1}(f) = S_{\alpha}(f).$$

Remark 3. The number h_i from (18), $i = \overline{1, r}$, represents the number of series of the Jordan basis (2) corresponding to eigenvalue α , formed with m_i vectors.

Next we determine, using expression (19), the basis $\{u_1^i, \dots, u_{q_i}^i\}$ in the space $S_{\alpha}^{m_i}(f)$, $i = \overline{1, r}$, and then, starting from basis $\{u_1^r, \dots, u_{q_r}^r\}$ of $S_{\alpha}^{m_r}(f)$ we first find a basis $\{v_1^{r-1}, \dots, v_{h_{r-1}}^{r-1}, u_1^r, \dots, u_{h_r}^r\}$ in the space $S_{\alpha}^{m_{r-1}}(f)$ (we have $S_{\alpha}^{m_r}(f) \subset S_{\alpha}^{m_{r-1}}(f)$, $q_r = h_r$, and $q_{r-1} = q_r + h_{r-1}$) then a basis of the form $\{v_1^{r-2}, \dots, v_{h_{r-2}}^{r-2}, v_1^{r-1}, \dots, v_{h_{r-1}}^{r-1}, u_1^r, \dots, u_{h_r}^r\}$ in $S_{\alpha}^{m_{r-2}}(f)$ (we have $S_{\alpha}^{m_{r-1}}(f) \subset S_{\alpha}^{m_{r-2}}(f)$ and $q_{r-2} = q_{r-1} + h_{r-2} = h_{r-2} + h_{r-1} + h_r$), and so on.

In this way we find a basis

$$(22) \quad \{v_1^1, \dots, v_{h_1}^1; v_1^2, \dots, v_{h_2}^2; \dots; v_1^{r-1}, \dots, v_{h_{r-1}}^{r-1}; v_1^r, \dots, v_{h_r}^r\}$$

in $S_{\alpha}^{m_1}(f)$, where we denoted $v_i^r = u_i^r$, $i = \overline{1, h_r}$.

Basis (22) has the property:

(A) The vector-system

$$(23) \quad \{v_1^i, \dots, v_{h_i}^i\}$$

is contained in the space $S_{\alpha}^{m_i}(f)$, $i = \overline{1, r}$.

Due to property (A) we can obtain, as in the proof of Lemma, the series of eigenvectors and associated vectors

$$(24) \quad \{w_{1k}^i, w_{2k}^i, \dots, w_{m_{ik}}^i\}, \quad k = \overline{1, h_i}, \quad i = \overline{1, r},$$

where $w_{1k}^i = v_k^i$, $k = \overline{1, h_i}$, corresponding to eigenvalue α .

Suppose we have found a set of series of vectors (24) in the described way. After that we consider the vector-system

$$(25) \quad \begin{aligned} &\{w_{11}^1, w_{21}^1, \dots, w_{m_{11}}^1; \dots; w_{1h_1}^1, w_{2h_1}^1, \dots, w_{m_{1h_1}}^1; \dots; w_{11}^r, \\ &w_{21}^r, \dots, w_{m_{r1}}^r; \dots; w_{1h_r}^r, w_{2h_r}^r, \dots, w_{m_{rh_r}}^r\}. \end{aligned}$$

Thus we arrive to

Theorem 2. Let be $f : X_n \rightarrow X_n$ a linear operator for which there is a Jordan basis in X_n of the form (2). Assume that conditions (10) and (18) are satisfied and a vector-system (25) is determined.

Then the vector-system

$$(25') \quad \left\{ w_{11}^1, w_{21}^1, \dots, w_{m_1 1}^1; \dots; w_{1h_1}^1, w_{2h_1}^1, \dots, w_{m_1 h_1}^1; \dots; w_{11}^r, \right. \\ \left. w_{21}^r, \dots, w_{m_r 1}^r; \dots; w_{1h_r}^r, w_{2h_r}^r, \dots, w_{m_r h_r}^r; a_1^{k+1}, \dots, a_{n_{k+1}}^{k+1}, \dots, a_1^p, \dots, a_{n_p}^p \right\},$$

is also a Jordan basis of f , in X_n .

Proof. Consider the vector system

$$(26) \quad \left\{ w_{11}^1, \dots, w_{1h_1}^1, w_{11}^2, \dots, w_{1h_2}^2, \dots, w_{11}^r; \dots; w_{1h_r}^r \right\}.$$

Because of the way in which the vector-system (25) has been obtained, (26) is a basis in space $S_\alpha(f)$, hence it is a linearly independent system of eigenvectors corresponding to eigenvalue α . If we also take the conditions $\alpha \neq \lambda_{k+1}, \dots, \lambda_p$ (we have (10)) into account, it follows that the vector system

$$(27) \quad \left\{ w_{11}^1, \dots, w_{1h_1}^1, w_{11}^2, \dots, w_{1h_2}^2, \dots, w_{11}^r; \dots; w_{1h_r}^r, a_1^{k+1}, \dots, a_1^p \right\}$$

is linearly independent [4]

Applying Theorem 2 from [6], we conclude that the vector-system (26) is linearly independent. We also have that, in (26), there are n vector ((26) is obtained from (2) by replacing the first $n_1 + n_2 + \dots + n_k$ vectors by $m_1 h_1 + m_2 h_2 + \dots + m_r h_r$, vectors, and from (18) it results $m_1 h_1 + m_2 h_2 + \dots + m_r h_r = n_1 + n_2 + \dots + n_k$).

Remark 4. In order to determine a vector system (25) it is necessary and sufficient to know only the pairs of numbers (m_1, h_1) , (m_2, h_2) , $\dots$, (m_r, h_r) from (18).

Connected with this fact we have

$$(28) \quad \text{rank}(A - \alpha I_n)^{k-1} - 2\text{rank}(A - \alpha I_n)^k + \text{rank}(A - \alpha I_n)^{k+1} = \\ = \begin{cases} h_i & \text{if } k = m_i, \quad i = \overline{1, r}, \\ 0 & \text{if } k = 1, 2, \dots, \quad k \neq m_1, m_2, \dots, m_r, \end{cases}$$

where A is the matrix of f with respect to basis (5) and $(A - \alpha I_n)^0 = I_n$ [7], [8].

This equality can be obtained by a direct calculus, using the known equality $\text{rank}(A - \alpha I_n)^k = \text{rank}(J - \alpha I_n)^k$, $k = 1, 2, \dots$, where J is the Jordan matrix (14).

It also results from (28) that the pairs of numbers (m_i, h_i) , $i = \overline{1, r}$ of (18) are uniquely determined for a linear operator f .

4. A Direct Method for Obtaining a Jordan Basis

Let $f : X_n \rightarrow X_n$ be a linear operator. Consider in X_n a basis (2) and then we determine the matrix A of f with respect to this basis and the characteristic

polynomial of f

$$(29) \quad p(\lambda) = \det(\lambda I_n - A).$$

It is known that there is a Jordan basis in X_n , corresponding to f , if and only if the polynomial (28) can be written in the form

$$(29') \quad p(\lambda) = (\lambda - \alpha_1)^{p_1} (\lambda - \alpha_2)^{p_2} \dots (\lambda - \alpha_q)^{p_q},$$

where $\alpha_i \in K$, $i = \overline{1, q}$, $\alpha_i \neq \alpha_j$, $i \neq j$, $i, j = \overline{1, q}$, $p_1 + p_2 + \dots + p_q = n$ [3], [6].

Suppose that polynomial (29) can be written as in (29'). Then, there is a Jordan basis in X_n of f , of the form

$$(30) \quad \left\{ a_{11}^{11}, a_{21}^{11}, \dots, a_{m_1^1}^{11}; \dots; a_{1h_1^1}^{11}, \dots, a_{m_1^1 h_1^1}^{11}; a_{11}^{12}, \dots, a_{m_2^1}^{12}; \dots; \right. \\ \left. a_{1h_2^1}^{12}, \dots, a_{m_2^1 h_2^1}^{12}; \dots; a_{11}^{1r_1}, \dots, a_{m_1^1}^{1r_1}; \dots; a_{1h_1^1}^{1r_1}, \dots, a_{m_1^1 h_1^1}^{1r_1}; \dots; \right. \\ \left. a_{11}^{q1}, \dots, a_{m_1^q}^{q1}; \dots; a_{1h_1^q}^{q1}, \dots, a_{m_1^q h_1^q}^{q1}; \dots; a_{11}^{qr_q}, \dots, a_{m_q^q}^{qr_q}; \dots; a_{1h_{r_q}^q}^{qr_q}, \dots, a_{m_q^q h_{r_q}^q}^{qr_q} \right\},$$

where $\{a_{1j}^{ik}, a_{2j}^{ik}, \dots, a_{m_k^i}^{ik}\}$ is a series of eigenvectors and associated vectors corresponding to eigenvalue α_i , $k = \overline{1, r_i}$, $j = \overline{1, h_k^i}$, $i = \overline{1, q}$.

We also have

$$(31) \quad 1 \leq m_1^i < \dots, m_{r_i}^i \leq p_i, \quad 1 \leq h_1^i, \dots, h_{r_i}^i \leq p_i, \\ m_1^i h_1^i + \dots + m_{r_i}^i h_{r_i}^i = p_i, \quad i = \overline{1, q}$$

[3], [5]. The pairs of numbers (m_j^i, h_j^i) $j = \overline{1, r_i}$, $i = \overline{1, q}$ which appear in (29) or the same in (30) can be calculated by means of equality (28).

Suppose that we found these pairs (m_j^i, h_j^i) , that it we have established the form (29) for a Jordan basis in X_n of the operator f .

Then, for every eigenvalue α_i , $i = \overline{1, q}$, we determine the series of eigenvectors and associated vectors

$$(24') \quad \{w_{1j}^{ik}, w_{2j}^{ik}, \dots, w_{m_k^i}^{ik}\}, \quad k = \overline{1, r_i}, \quad j = \overline{1, h_k^i},$$

in the way in which series (24) has been obtained.

Then we apply Theorem 2, first relatively to the series corresponding to eigenvalue α_1 from (29) and (24'). So, we obtain a new Jordan basis of f . In this new basis we replace in similar way the series corresponding to eigenvalue α_2 and so on. Finally we obtain the vector system

$$(30') \quad \left\{ w_{11}^{11}, w_{21}^{11}, \dots, w_{m1}^{11}; \dots; w_{1h_q}^{qr_q}, \dots, w_{m_q h_q}^{qr_q} \right\},$$

which is a Jordan basis in X_n of operator f .

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REFERENCES

1. Corduneanu A., *Ecuații diferențiale cu aplicații în electrotehnică*. Ed. "Facla", Timișoara, 1981.
2. Corduneanu A., *A Direct Method for the Jordan Form of the Fourth Order Matrix*. Bul. Inst. Polit. Iași, **XXXIX (XLIII)**, 1 - 4, s. I (1993).
3. Filipov A. F., *Kratkoe dokazatel'stvo teoremy o privedenii matricy k Jordanovoj forme*. Vest. Mosk. Universiteta, 2, 18-19 (1971).
4. Ghelfand I. M., *Lekcii po lineinoj algebre*. Izd. Nauka, Moskva, 1980.
5. Onciulescu I., *Complemente de algebră liniară și aplicații*. Univ. Tehnică "Gh. Asachi", Iași, 1993.
6. Onciulescu I., *Properties Concerning a Direct Method for a Jordan Basis of a Linear Operator*. Bul. Inst. Polit. Iași, **XL (XLIV)**, 1 - 4, s. I (1994).
7. Pop I., *Curs de algebră*. Univ. "Al. I. Cuza", Iași, 1978.
8. Proskurjakov I. V., *Problems in Linear Algebra*. MIR Publisher, Moscow, 1978.

O METODĂ DIRECTĂ PENTRU DETERMINAREA UNEI BAZE JORDAN ÎNTR-UN SPAȚIU VECTORIAL n -DIMENSIONAL

(Rezumat)

Pe baza faptului că $S_{\lambda}^k(f)$ dat de (4) este subspațiu vectorial al spațiului vectorial $S_{\lambda}(f)$ (3), se determină dimensiunea spațiului $S_{\lambda_i}^{n_i}(f)$, cu λ_i, n_i precizați prin (2), și utilizând alte proprietăți specifice seriilor de vectori și de vectori asociați [5], [6], se pun bazele unei metode directe de determinare a unei baze Jordan de forma (2).

THE CONSTRUCTION OF A JORDAN BASIS

BY

ARIADNA LUCIA PLETEA and MARIA BECEANU

Abstract. An algorithm for a direct method to obtain a Jordan basis in an n -dimensional vector space is developed.

Key words: Jordan basis, vector space, method of completion, algorithm.

1. Preliminaires

Let $f : V_n \rightarrow V_n$ be an endomorphism of the n -dimensional vector space V_n over a commutative field K . We suppose that the characteristic polynomial of f is given by

$$(1) \quad p(\lambda) = (\lambda - \alpha_1)^{p_1} (\lambda - \alpha_2)^{p_2} \dots (\lambda - \alpha_q)^{p_q},$$

$$\alpha_i \in K, \quad i = \overline{1, q}; \quad \alpha_i \neq \alpha_j; \quad i \neq j; \quad i, j = \overline{1, q}; \quad p_1 + p_2 + \dots + p_q = n.$$

It is known that there exists a Jordan basis in V_n , corresponding to f [1].

In the following we present an algorithm for a Jordan basis in V_n . We consider in this space a basis $\{e_1, e_2, \dots, e_n\}$ and we denote by A the matrix of the endomorphism f relative to this basis.

If $\alpha \in K$ is an eigenvalue of f , then the system of vectors $\{a_1, a_2, \dots, a_k\}$ of V_n satisfying

$$f(a_1) = \alpha a_1, \quad f(a_2) = \alpha a_2 + a_1, \dots, \quad f(a_k) = \alpha a_k + a_{k-1}, \quad a_1 \neq \theta$$

is called a series of eigenvector and associated vectors.

A Jordan basis in V_n , corresponding to the operator f , is a basis in V_n of the form

$$(2) \quad \{a_1^1, \dots, a_{n_1}^1, a_1^2, \dots, a_{n_2}^2, \dots, a_1^p, \dots, a_{n_p}^p\}, \quad n_1 + n_2 + \dots + n_p = n,$$

where $\{a_1^i, a_2^i, \dots, a_{n_i}^i\}$, $i = \overline{1, p}$, are series of eigenvectors and associated vectors. We consider the sets

$$S_\alpha(f) = \{v | v \in V_n, f(v) = \alpha v\},$$

$$(3) \quad S_\alpha^k(f) = \{v | v \in S_\alpha(f), \exists v_1, \dots, v_{k-1} \in V_n\} \text{ so that}$$

$$f(v_1) = \alpha v_1 + v, f(v_2) = \alpha v_2 + v_1, \dots, f(v_{k-1}) = \alpha v_{k-1} + v_{k-2}\},$$

where $\alpha \in K$ and $k \geq 2$ is an integer.

Assume that α is an eigenvalue of f with its multiplicity p .

Theorem 1. *The vector $v \in S_\alpha^k(f)$, $k \in \mathbb{N}$, $k \geq 2$, if and only if $v \in S_\alpha(f)$ and there exists $w \in V_n$ so that*

$$(4) \quad (f - \alpha\varepsilon)^{k-1}w = v,$$

where ε is the identical operator on the vector space V_n .

Proof. Suppose that $v \in S_\alpha^k(f)$. It results that $(f - \alpha\varepsilon)v = \theta$, so that there exist $(k-1)$ vectors $v_1, v_2, \dots, v_{k-1} \in V_n$ so that

$$(f - \alpha\varepsilon)v_1 = v, (f - \alpha\varepsilon)v_2 = v_1, \dots, (f - \alpha\varepsilon)v_{k-1} = v_{k-2}.$$

We denote $v_{k-1} = w$. Applying the transformation $(f - \alpha\varepsilon)^{k-2}$ to both sides of the equality $(f - \alpha\varepsilon)w = v_{k-2}$, we get $(f - \alpha\varepsilon)^{k-1}w = v$.

Conversely, if $v \in S_\alpha(f)$ so that there exists $w \neq \theta$ for which we have the relation $(f - \alpha\varepsilon)^{k-1}w = v$ we set $w = v_{k-1}$. We obtain the relations $(f - \alpha\varepsilon)v_1 = v$, $(f - \alpha\varepsilon)v_2 = v_1, \dots, (f - \alpha\varepsilon)v_{k-1} = v_{k-2}$, $v_i \neq \theta$, $i = \overline{1, k-1}$. Thus $v \in S_\alpha^k(f)$.

We remark that the vectors $\{v, v_1, v_2, \dots, v_{k-1}\}$ are linearly independent and form a series of an eigenvector and associated vectors.

Theorem 2. *Let $\mathcal{B} = \{u_1, u_2, \dots, u_s\}$ be a basis in the subspace $\ker(f - \alpha\varepsilon)^k$, $k \in \mathbb{N}$, $k \geq 2$, where $s = \text{nullity}(f - \alpha\varepsilon)^k$ and $L = \{z_1, z_2, \dots, z_s\}$ with*

$$(5) \quad z_i = (f - \alpha\varepsilon)^{k-1}u_i, \quad i = \overline{1, s}.$$

Then $v \in S_\alpha^k(f)$ if and only if $v \in [L]$, where $[L]$ is the vector subspace spanned by the system of vectors $\{z_1, z_2, \dots, z_s\}$.

Proof. Suppose that $v \in S_\alpha^k(f)$. Using Theorem 1 there exists a vector $w \neq \theta$, $w \in V_n$ so that $(f - \alpha\varepsilon)^{k-1}w = v$. Applying the endomorphism $(f - \alpha\varepsilon)$ to this relation we get $w \in \ker(f - \alpha\varepsilon)^k$ and there exist $\mu_i \in K$, $i = \overline{1, s}$ so that $w = \sum_{i=1}^s \mu_i u_i$ and $v = (f - \alpha\varepsilon)^{k-1}w = \sum_{i=1}^s \mu_i (f - \alpha\varepsilon)^{k-1}u_i$ and $v = \sum_{i=1}^s \mu_i z_i$. Thus $v \in [L]$.

Conversely, let be $v \in [L]$. Obviously $v = \sum_{i=1}^s \mu_i z_i$ and $v = \sum_{i=1}^s \mu_i (f - \alpha\varepsilon)^{k-1} u_i = (f - \alpha\varepsilon)^{k-1} (\sum_{i=1}^s \mu_i u_i)$. So there exists $w = \sum_{i=1}^s \mu_i u_i \in \ker(f - \alpha\varepsilon)^k$, $w \neq \theta$ so that $(f - \alpha\varepsilon)^{k-1} w = v$ and $(f - \alpha\varepsilon)^{k-j} w = v_{j-1}$, $j = \overline{2, k}$, where $(f - \alpha\varepsilon)^0 = \varepsilon$. This means that $z \in S_\alpha^k(f)$.

Remark 1. The subspace $S_\alpha^k(f)$ is spanned by the system of vectors $\{z_1, z_2, \dots, z_s\}$ constructed using the relations (5). A basis in $S_\alpha^k(f)$ is a maximum linearly independent subset of this system of vectors. We use this result to construct bases in the spaces $S_\alpha^k(f)$.

Remark 2. Let $v \in V_n$ be so that $v = \sum_{i=1}^s \rho_i e_i$, $\rho_i \in K$, $i = \overline{1, n}$. Then $v \in S_\alpha^k(f)$ if and only if the algebraic system $(A - \alpha I_n)^{k-1} X = Y$ is consistent in K , where we denote by $X = [\nu_1, \nu_2, \dots, \nu_n]^T$, $Y = [\rho_1, \rho_2, \dots, \rho_n]^T$ and I_n unit matrix of order n .

Let be $d = \text{nullity}(f - \alpha\varepsilon)$, $1 \leq d \leq p$. We remark that d is equal to the number of series of eigenvectors and associated vectors corresponding to the eigenvalue α . The number of series of eigenvectors and associated vectors of length m_i is obtained using the relation

$$(6) \quad \begin{aligned} \varphi(j, \alpha) &= \text{rank}(A - \alpha I_n)^{j-1} - 2\text{rank}(A - \alpha I_n)^j + \text{rank}(A - \alpha I_n)^{j+1} = \\ &= \begin{cases} h_i, & \text{if } j = m_i, i = \overline{1, r}; \\ 0, & \text{if } j = 1, 2, \dots; j \neq m_1, m_2, \dots, m_r, \end{cases} \end{aligned}$$

where r is the number of distinguished in length series of eigenvectors and associated vectors corresponding to the eigenvalue α [3].

The numbers $h_i, i = \overline{1, r}$, from (6) represent the numbers of series of eigenvectors and associated vectors, formed with exact m_i vectors, $i = \overline{1, r}$, corresponding to the eigenvalue α . We suppose that $1 \leq m_1 < m_2 < \dots < m_r \leq p$.

We have $\sum_{i=1}^r h_i = d$ and $\sum_{i=1}^r h_i m_i = p$. The pairs (m_i, h_i) are uniquely determined for an endomorphism.

Remark 3. As a consequence of Theorem 1 [2] we have $\dim S_\alpha^{m_i}(f) = q_i$, $q_i = d - (h_1 + h_2 + \dots + h_{i-1})$, $i = \overline{2, r}$, $r \leq d$, and $\dim S_\alpha^{m_1}(f) = d$.

Moreover $S_\alpha^{m_i}(f) \subset S_\alpha^{m_j}(f)$ for $m_i > m_j$ and $i, j = \overline{1, r}$.

2. The Construction of a Jordan Basis

We construct, using Theorem 2, the bases $\{v_1^i, v_2^i, \dots, v_{q_i}^i\}$ in the spaces $S_\alpha^{m_i}(f)$, $i = \overline{1, r}$.

Then we start from this initial basis of the space $S_\alpha^{m_r}(f)$, and using the basis before constructed in $S_\alpha^{m_{r-1}}(f)$, we obtain a new one in the space $S_\alpha^{m_{r-1}}(f)$ denoted

$\{v_1^{r-1}, v_2^{r-1}, \dots, v_{h_{r-1}}^{r-1}; v_1^r, v_2^r, \dots, v_{q_r}^r\}$ using the method of completion of a linearly independent system of vectors up to a basis and so on (we have $S_\alpha^{m_r}(f) \subset S_\alpha^{m_{r-1}}(f)$, $q_r = h_r$, $q_{r-1} = q_r + h_{r-1}$).

In this way we find a new basis

$$\{v_1^1, v_2^1, \dots, v_{h_1}^1; v_1^2, v_2^2, \dots, v_{h_2}^2; \dots; v_1^{r-1}, v_2^{r-1}, \dots, v_{h_{r-1}}^{r-1}; v_1^r, v_2^r, \dots, v_{q_r}^r\}$$

in $S_\alpha^{m_1}(f)$. For these vectors we construct the series of associated vectors denoted

$$(7) \quad \{w_{1k}^i, w_{2k}^i, \dots, w_{m_{ik}}^i\}, \quad k = \overline{1, h_i}, \quad i = \overline{1, r},$$

where

$$(8) \quad \begin{aligned} w_{1k}^i &= v_k^i, & k &= \overline{1, h_i}, & i &= \overline{1, r}; \\ w_{jk}^i &= (f - \alpha\varepsilon)^{m_i-j} w_{m_{ik}}^i, & j &= \overline{2, m_i}, & i &= \overline{1, r}. \end{aligned}$$

We remark that the system of vectors

$$\begin{aligned} S &= \{w_{11}^1, w_{21}^1, \dots, w_{m_{11}}^1; \dots; w_{1h_1}^1, w_{2h_1}^1, \dots, w_{m_{1h_1}}^1; \dots; w_{11}^r, \\ &w_{21}^r, \dots, w_{m_{r1}}^r; \dots; w_{1h_r}^r, w_{2h_r}^r, \dots, w_{m_{rh_r}}^r; a_1^{d+1}, \dots, a_1^p, \dots, a_{n_p}^p\} \end{aligned}$$

form a Jordan basis in V_n [2].

Remark 4. For every eigenvalue $\alpha_i, i = \overline{1, q}$, we repeat this construction.

3. The Steps of the Algorithm

1. Consider in V_n a basis and compute the matrix A of the endomorphism f relative to this basis.

2. Compute the characteristic polynomial (1).

3. For $i = \overline{1, q}$ and for the corresponding eigenvalue α_i :

a) Compute $d_i = \text{nullity}(A - \alpha_i I_n), i = \overline{1, q}$.

b) Compute, using the relation (6)

$$(m_\ell^i, h_\ell^i; r_i), \quad 1 \leq m_\ell^i \leq p_i, \quad 1 \leq h_\ell^i \leq k_i,$$

$$1 \leq r_i \leq k_i, \quad \ell = \overline{1, r_i}, \quad 1 \leq m_1^i < m_2^i < \dots < m_{r_i}^i \leq p_i,$$

$$\sum_{\ell=1}^{r_i} h_\ell^i m_\ell^i = p_i \quad \text{and} \quad \sum_{\ell=1}^{r_i} h_\ell^i = d_i, \quad i = \overline{1, s}.$$

The computation is finished when

$$m_1^i h_1^i + m_2^i h_2^i + \dots + m_{r_i}^i h_{r_i}^i = p_i.$$

c) For every $\ell = \overline{1, r_i}$, compute a basis in $\ker(A - \alpha_i I_n)^{m_i}$ and a new basis in $S_\alpha^{m_1}(f)$.

d) Using the relations (8), we construct the corresponding associated vectors.

We illustrate this algorithm by means of two examples of linear operators defined on the vector space R^{11} .

Example 1. Let the endomorphism f be defined on the vector space R^{11} by

$$f(x) = \left(\frac{3}{2}x_1 + 5x_2 + \frac{7}{2}x_3 + \frac{13}{2}x_4 + x_5 - 9x_6 - \frac{13}{2}x_7 + \frac{9}{2}x_8 - 8x_{10} - 2x_{11}, \right.$$

$$\left. \frac{1}{2}x_1 + 3x_2 + \frac{1}{2}x_3 + \frac{3}{2}x_4 - x_6 - \frac{3}{2}x_7 + \frac{1}{2}x_8 - 2x_{10}, \right.$$

$$\left. -x_2 + x_3 - 2x_4 + 2x_5 + 2x_7 - x_8 + 3x_{10}, \right.$$

$$\left. -x_1 + 3x_2 + 2x_3 + 6x_4 + 2x_5 - 6x_6 - 4x_7 + 3x_8 - 5x_{10} - x_{11}, \right.$$

$$\left. -x_1 + 3x_2 + 2x_3 + 4x_4 + 4x_5 - 7x_6 - 4x_7 + 3x_8 - 5x_{10} - x_{11}, \right.$$

$$\left. 2x_6, 2x_2 + 2x_3 + 3x_4 - 3x_6 - x_7 + 2x_8 - 4x_{10} - x_{11}, \right.$$

$$\left. \frac{5}{2}x_1 - 3x_2 - \frac{3}{2}x_3 - \frac{9}{2}x_4 - 5x_5 + 10x_6 - \frac{9}{2}x_7 - \frac{3}{2}x_8 + 6x_{10}, \right.$$

$$\left. \frac{3}{2}x_1 - x_2 - \frac{1}{2}x_3 - \frac{3}{2}x_4 - 2x_5 + 4x_6 + \frac{3}{2}x_7 - \frac{3}{2}x_8 + 2x_9 + 2x_{10}, \right.$$

$$\left. \frac{1}{2}x_1 - \frac{1}{2}x_3 - \frac{1}{2}x_4 + x_6 + \frac{1}{2}x_7 - \frac{1}{2}x_8 + 3x_{10}, \right.$$

$$\left. x_1 - 3x_2 - 2x_3 - 5x_4 + 5x_6 + 5x_7 - 3x_8 + 7x_{10} + 2x_{11} \right).$$

We consider the canonical basis $\{e_1, e_2, \dots, e_{11}\}$ and let A be the matrix of the operator in this basis. The characteristic polynomial is $p(\lambda) = (\lambda - 2)^{11}$.

For $\lambda = 2$ we compute $\text{rank}(A - 2I_{11}) = 5$ and so we have $d_1 = 6$. Let the bases in the null spaces of the operators $(f - 2\varepsilon)^j, j = 1, 2, 3$ be

$$ns1 = \{(0, 1, 1, -1, 0, 0, 0, 0, 0, 0, 1), (0, -1, 0, 2, 0, 0, 0, 0, 0, 1, 0),$$

$$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0), (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 1, 0, 0, 0),$$

$$(0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0)\}$$

$$\begin{aligned}
ns2 = \{ & (-\frac{2}{5}, \frac{1}{5}, 0, 0, 0, 0, 0, 0, 0, 0, 1), (\frac{2}{5}, \frac{9}{5}, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0), \\
& (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0), (\frac{3}{5}, -\frac{4}{5}, 0, 0, 0, 0, 0, 1, 0, 0, 0), \\
& (\frac{1}{5}, \frac{7}{5}, 0, 0, 0, 0, 1, 0, 0, 0, 0), (-\frac{14}{5}, \frac{7}{5}, 0, 0, 0, 1, 0, 0, 0, 0, 0), \\
& (2, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0) (-\frac{1}{5}, -\frac{7}{5}, 0, 1, 0, 0, 0, 0, 0, 0, 0), \\
& (\frac{1}{5}, -\frac{3}{5}, 1, 0, 0, 0, 0, 0, 0, 0, 0)\}
\end{aligned}$$

$$ns3 = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}.$$

Using the relation (6) we compute

$$\begin{aligned}
h_1 &= \varphi(1, 2) = 11 - 2.6 + 2 = 1, & m_1 &= 1, \\
h_2 &= \varphi(2, 2) = 6 - 2.2 + 0 = 2, & m_2 &= 2, \\
h_3 &= \varphi(3, 2) = 2 - 2.0 + 0 = 2, & m_3 &= 3.
\end{aligned}$$

We remark that $m_1 h_1 + m_2 h_2 + m_3 h_3 = 1 + 4 + 6 = 11$, so that the computation is finished.

Using the vectors from $ns3$ and the relation (5) we obtain, conforming Theorem 2, the system of vectors

$$\begin{aligned}
& \{(\frac{1}{2}, -1, -\frac{3}{2}, 0, 0, 0, 0, 1, -\frac{1}{2}, -\frac{1}{2}, -1), (1, 3, 7, 0, 0, 0, 0, 2, 4, 4, 8), \\
& (\frac{1}{2}, 2, \frac{9}{2}, 0, 0, 0, 0, 1, \frac{5}{2}, \frac{5}{2}, 5), (-\frac{3}{2}, 4, \frac{19}{2}, 0, 0, 0, 0, 3, \frac{11}{2}, \frac{11}{2}, 11), \\
& (-1, 2, 3, 0, 0, 0, 0, -2, 1, 1, 2), (0, -7, -14, 0, 0, 0, 0, 0, -7, -7, -14), \\
& (-\frac{1}{2}, 3, \frac{13}{2}, 0, 0, 0, 0, 1, \frac{7}{2}, \frac{7}{2}, 7), (-2, -5, -12, 0, 0, 0, 0, -4, -7, -7, -14), \\
& (0, -1, -2, 0, 0, 0, 0, 0, -1, -1, -2)\}.
\end{aligned}$$

A basis in the space $S_2^3(f)$ is ($\dim S_2^3(f) = 2$)

$$\begin{aligned}
(9) \quad & \{(0, -1, -2, 0, 0, 0, 0, 0, -1, -1, -2), \\
& (-2, -5, -12, 0, 0, 0, 0, 0, -4, -7, -7, -14)\}
\end{aligned}$$

and we construct the associated vectors for these eigenvectors, using the relation (8)

$$\begin{aligned}
& \{(0, -1, -2, 0, 0, 0, 0, 0, -1, -1, -2), (-2, 0, 0, -1, -1, 0, -1, 0, 0, 0, 0), \\
& (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)\}
\end{aligned}$$

Example 2. Let the endomorphism f of the form be

$$f(x) = \left(\frac{7}{2}x_1 + 2x_2 + \frac{3}{2}x_3 + \frac{3}{2}x_4 + x_5 - 5x_6 - \frac{3}{2}x_7 + \frac{3}{2}x_8 - 2x_{10} - 2x_{11}, \right.$$

$$\left. \frac{9}{2}x_1 - 7x_2 - \frac{11}{2}x_3 - \frac{31}{2}x_4 - 10x_5 + 24x_6 + \frac{33}{2}x_7 - \frac{23}{2}x_8 + 18x_{10} + 6x_{11}, \right.$$

$$2x_1 - 12x_2 - 5x_3 - 20x_4 - 5x_5 + 22x_6 + 21x_7 - 12x_8 + 23x_{10} + 8x_{11},$$

$$-2x_1 - 2x_3 + x_4 + 4x_5 - 4x_6 + 4x_7 + 2x_{10} + 2x_{11},$$

$$x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 - x_6 - 2x_7 + x_8 - 3x_{10} - 2x_{11},$$

$$2x_1 - 2x_2 - 2x_4 - 4x_5 + 10x_6 + 2x_7 - 2x_8 + 2x_{10},$$

$$-x_1 + x_2 + x_4 + x_5 - 3x_6 + 4x_7 + x_8 - x_{10},$$

$$\frac{9}{2}x_1 - 5x_2 - \frac{3}{2}x_3 - \frac{13}{2}x_4 - 9x_5 + 16x_6 + \frac{13}{2}x_7 - \frac{3}{2}x_8 + 8x_{10},$$

$$\frac{7}{2}x_1 - 10x_2 - \frac{13}{2}x_3 - \frac{33}{2}x_4 - 7x_5 + 21x_6 + \frac{33}{2}x_7 - \frac{21}{2}x_8 + 2x_9 + 20x_{10} + 6x_{11},$$

$$\frac{5}{2}x_1 - 9x_2 - \frac{13}{2}x_3 - \frac{31}{2}x_4 - 5x_5 + 18x_6 + \frac{31}{2}x_7 - \frac{19}{2}x_8 + 21x_{10} + 6x_{11},$$

$$\left. \frac{3}{2}x_1 - \frac{15}{2}x_2 - 5x_3 - \frac{25}{2}x_4 - 3x_5 + \frac{27}{2}x_6 + \frac{25}{2}x_7 - \frac{15}{2}x_8 + \frac{29}{2}x_{10} + 8x_{11} \right)$$

We consider the canonical basis $\{e_1, e_2, \dots, e_{11}\}$ and let A be the matrix of the operator with respect to this basis. The characteristic polynomial is $p(\lambda) = (\lambda - 2)^3(\lambda - 4)^8$.

For $\lambda = 2$ we have $d_1 = 2$ and for $\lambda = 4$, $d_2 = 3$. For $\lambda = 2$ we have $h_1 = 1$, $m_1 = 1$, $h_2 = 1$, $m_2 = 2$.

We remark that $m_1h_1 + m_2h_2 = 3$, so that the computation is finished.

For $\lambda = 4$ we obtain, in the same way, $h_1 = 1$, $m_1 = 1$, $h_2 = 1$, $m_2 = 3$, $h_3 = 1$, $m_3 = 4$.

A basis in the space $\ker(A - 2I_{11})^2$ is composed from vectors

$$\{(0, 0, 2, 2, 0, 0, 0, 0, 0, 2, 1), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0), \\ (0, 2, 0, -1, 1, 1, 0, 1, 0, 0, 0)\}$$

and one in $S_2^2(f)$, $\dim S_2^2(f) = 1$, is formed of the eigenvector $\{(0, 0, 1, 1, 0, 0, 0, 0, 0, 1, \frac{1}{2})\}$. We completed this basis up to a basis in $S_2^1(f)$ and we obtain

$$\{(0, 0, 1, 1, 0, 0, 0, 0, 0, 1, \frac{1}{2}), (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0)\}.$$

The associated vector for the vector from $S_2^2(f)$ is $\{(0, 2, 0, -1, 1, 1, 0, 1, 0, 0, 0)\}$. We consider a basis in $\ker(A - 4I_{11})^4$ formed by the vectors

$$\begin{aligned} &\{(1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1), (1, 2, 0, 0, 0, 0, 0, 0, 1, 1, 0), \\ &(0, -1, 0, 0, 0, 0, 0, 10, 0, 0, 0), (1, 2, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0), \\ &(-3, 3, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0), (2, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0) \\ &(-1, -2, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0), (-1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)\} \end{aligned}$$

and then we obtain a basis in $S_4^4(f)$ of the form $\{(-1, 0, -1, 0, 0, 0, 0, -2, -1, -1, -1)\}$.

Using the same method we obtain bases in the spaces $S_4^3(f)$ and $S_4^1(f)$. These are

$$\{(-\frac{1}{2}, 0, -\frac{1}{2}, 0, 0, 0, 0, -1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (\frac{9}{2}, 0, \frac{1}{2}, 2, 2, 0, 2, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$$

and respectively

$$\begin{aligned} &\{(-\frac{1}{2}, 0, -\frac{1}{2}, 0, 0, 0, 0, -1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (\frac{9}{2}, 0, \frac{1}{2}, 2, 2, 0, 2, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ &(-4, 2, 1, 0, 0, 2, 0, 0, 0, 0, 0)\} \end{aligned}$$

The Jordan basis is of the form

$$\begin{aligned} &\{(-1, 0, -1, 0, 0, 0, 0, -2, -1, -1, -1), (\frac{1}{2}, -1, \frac{3}{2}, 0, 0, 0, 0, 2, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}), \\ &(-\frac{1}{2}, -\frac{1}{2}, -2, 0, 0, 0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -2), (1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1), \\ &(\frac{9}{2}, 0, \frac{1}{2}, 2, 2, 0, 2, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{5}{2}, \frac{5}{2}, 3, -2, -2, 0, 0, \frac{5}{2}, \frac{3}{2}, \frac{3}{2}, 1), \\ &(-4, 2, 1, 0, 0, 2, 0, 0, 0, 0, 0), (5, -2, -3, 2, 0, -2, 0, 0, 1, 1, 0), \\ &(0, 0, 1, 1, 0, 0, 0, 0, 0, 1, \frac{1}{2}), (0, 2, 0, -1, 1, 1, 0, 1, 0, 0, 0), \\ &(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0)\} \end{aligned}$$

and the Jordan form is

$$\begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

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REFERENCES

1. Horn R., Johnson C., *Matrix Analysis*. Cambridge University Press, 1986.
2. Onciulescu I., *A Direct Method for a Jordan Basis in an n-dimensional Vector Space*. Bul. Inst. Polit., Iași, **XLIII (XLVII)**, 3 - 4, s. I (1997).
3. Proskuriakov I. V., *Problems in Linear Algebra*. MIR Publisher, Moscow, 1978.

CONSTRUCȚIA UNEI BAZE JORDAN

(Rezumat)

Se elaborează un algoritm de construcție directă a unei baze Jordan într-un spațiu vectorial n -dimensional. Două exemple ilustrează eficacitatea algoritmului.

POINTS FIXES COMMUNS ET POINTS DE COÏNCIDENCE POUR UN OPÉRATEUR ET UNE APPLICATION MULTIVOQUE FAIBLEMENT COMMUTATIFS

PAR

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Abstract. We prove two common fixed point theorems on weakly commuting mappings and multifunctions in a complete metric space.

Key words: fixed point theorem, multifunction, weakly commutativity, contractive inequality, complete metrical space.

1. Introduction

En partant des résultats de S. S e s s a [9] (1982) et G. J u n g c k [2] (1976) pour des opérateurs contractifs faiblement commutatifs, des résultats sur les points de coïncidence des opérateurs et des applications multivoques de K. P. R a o [8] (1987) et des résultats de points fixes communs pour un opérateur et une application multivoque faiblement commutatifs de H. K a n e k o [5] (1988), nous démontrons dans cette Note quelques théorèmes de point de coïncidence et de point fixe commun pour un opérateur et une application multivoque faiblement commutatifs définis sur des espaces métriques complets, mais en employant d'autres types de conditions contractives, conditions qui nous ont été suggérées par les théorèmes obtenus par R. K. J a i n et S. P. D i x i t [3] (1986) et par Nicoleta N e g o e s c u [6], [7] (1989, 1996).

Soient (X, d) un espace métrique, $K(X)$ l'ensemble de tous sousensembles compacts, nonvides de X , $D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ (où A ou B peut être un ensemble formé d'un seul point), et la métrique de Hausdorff définie sur $K(X)$, induite par la métrique d (v. [1]):

$$H(A, B) = \max \left\{ \sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A) \right\}$$

Un sousensemble nonvide S de X s'appelle proximal si pour tout $x \in X$ il existe $s \in S$ tel que $d(x, s) = D(x, S)$. Tout sousensemble compact de X est un ensemble proximal (v. [4]).

Soient l'opérateur $f : X \rightarrow X$ et l'application multivoque $T : X \rightarrow K(X)$, où (X, d) est un espace métrique. On dit que f, T sont commutatifs si $fTx \subseteq Tf x$, $\forall x \in X$ et on dit que f, T sont faiblement commutatifs si

$$\forall x \in X, fTx \in K(X) \text{ et } H(Tfx, fTx) \leq D(fx, Tx).$$

Un point $x \in X$ s'appelle point de coïncidence de f et T si $fx \in Tx$; x est un point fixe de f si $x = fx$; x est un point fixe de T si $x \in Tx$ et $x \in X$ est un point fixe commun de f et T si $fx = x \in Tx$.

2. Les principaux résultats

L'extension de la notion de commutativité faible pour les applications multivoques a été réalisée par H. K a n e k o ([5], 1988) en partant des résultats importants pour les applications $f : X \rightarrow X$ faiblement commutatives obtenus par S. S e s s a ([9], 1982) et G. J u n g c k ([2], 1976). Aussi dans sa note [9] S. S e s s a a démontré que deux applications commutatives sont faiblement commutatives, mais l'affirmation réciproque n'est pas vraie.

Maintenant nous démontrons les résultats suivants:

T h é o r è m e 1. Soient (X, d) un espace métrique complet et $f : X \rightarrow X$, $T : X \rightarrow K(X)$ deux applications continues. Si on a:

- (i) $T(X) \subseteq f(X)$,
 - (ii) T et f sont faiblement commutatives,
 - (iii) $H^2(Tx, Ty) \leq aD(fx, Tx)D(fy, Ty) + bD(fx, Ty)D(fy, Tx)$ pour tous $x, y \in X$, $x \neq y$, $x \in [0, 1)$ et $b \in \mathbb{R}_+$,
- alors f et T ont un point de coïncidence $t \in T$ ($ft \in Tt$).

De plus, si

- (iv) $\begin{cases} \text{ou } ft \neq ft^2 \text{ implique } ft \cap Tt = \emptyset, \\ \text{ou } ft \in Tt \text{ implique que } \lim_{n \rightarrow \infty} f^n t \text{ existe,} \end{cases}$
- alors T et f ont un point fixe commun $t \in X$.

D é m o n s t r a t i o n. Soit $x_0 \in X$ un point arbitraire de X et on choisit $x_1 \in X$ tel que $fx_1 \in Tx_0$ (cela est possible parce que $Tx_0 \subseteq f(X)$). De la définition de H il existe un point $y_1 \in Tx_1$ tel que $d(y, fx_1) \leq H(Tx_0, Tx_1)$. Car $Tx_1 \subseteq f(X)$, soit $x_0 \in X$ tel que $y_1 = fx_2$.

En général, après le choix de $x_n \in X$, on prend $x_{n+1} \in X$ tel que $x_{n+1} \neq x_n$ et

$y_n = fx_{n+1} \in Tx_n$ et $d(y_n, fx_n) \leq H(Tx_n, Tx_{n+1})$, $\forall n > 1$. Alors on a

$$d^2(fx_n, fx_{n+1}) \leq H^2(Tx_{n-1}, Tx_n) \leq aD(fx_{n-1}, Tx_{n-1})D(fx_n, Tx_n) + \\ + bD(fx_{n-1}, Tx_n)D(fx_n, Tx_{n-1})$$

ou

$$d^2(fx_n, fx_{n+1}) \leq ad(fx_{n-1}, fx_n)d(fx_n, fx_{n+1})$$

($D(fx_n, Tx_{n-1}) = 0$ car $fx_n \in Tx_{n-1}$ et Tx_{n-1}, Tx_n sont des ensembles proximinales).

Car $d(fx_n, fx_{n+1}) > 0$ de la construction de la suite $\{fx_n\}$, alors on obtient $d(fx_n, fx_{n+1}) \leq ad(fx_{n-1}, fx_n)$ pour tout $n \in \mathbb{N}$ (où $a \in [0, 1]$). Donc la suite $d(fx_n, fx_{n+1}) \searrow 0$ et il suit que $\{fx_n\}$ est une suite de Cauchy. Comme (X, d) est un espace métrique complet, il suit que $\{fx_n\}$ est une suite convergente à $t \in X$.

Nous montrons d'abord que $ft \in Tt$ (t est un point de coïncidence de f et T). En employant la continuité de l'application f et de l'application multivoque T et aussi leur commutativité faible, il suit:

$$D(ft, Tt) \leq d(ft, ffx_{n+1}) + D(ffx_{n+1}, Tt) \leq d(ft, ffx_{n+1}) + H(fTx_n, Tt) \leq \\ \leq d(ft, ffx_{n+1}) + H(fTx_n, Tfx_{n+1}) + H(Tfx_{n+1}, Tt) \leq \\ \leq d(ft, ffx_{n+1}) + D(fx_{n+1}, Tx_n) + H(Tfx_{n+1}, Tt).$$

On voit que $d(ft, ffx_{n+1}) \rightarrow 0$ et $H(Tfx_{n+1}, Tt) \rightarrow 0$ pour $n \rightarrow \infty$ et $D(fx_{n+1}, Tx_n) = 0$ car $fx_{n+1} \in Tx_n$, $\forall n \in \mathbb{N}$ (à cause de la construction de fx_{n+1}). Ainsi on obtient que $D(ft, Tt) = 0$ et, parce que l'ensemble Tt est fermé, il suit que $ft \in Tt$, donc z est un point de coïncidence pour f et T .

Pour démontrer que t est un point fixe commun de f et T on part des conditions (iv) et on obtient (en employant un raisonnement analogue de celui de H. K a n e k o [5]) les cas suivants:

a) S'il est vrai que la condition $ft \neq ft^2$ implique $ft \cap Tt = \emptyset$, alors $ft \in Tt$ de (i), (ii) et (iii) et donc $ft = ft^2$. Alors $D(ft, Tft) \leq d(ft, f^2t) + D(f^2t, Tft) = D(f^2t, Tft) \leq H(Tft, fTt) \leq D(ft, Tt) = 0$, ce qui implique que ft est un point fixe commun de f et T .

b) S'il est vrai que la condition $ft \in Tt$ implique $\lim_{n \rightarrow \infty} f^n t$ existe, alors du raisonnement antérieur $D(f^2t, Tft) = 0$, c'est-à-dire $ft \in Tft$.

Par induction et en employant la commutativité faible de f et T , on obtient $f^n t \in T f^n t$. Mais, il existe $\lim_{n \rightarrow \infty} f^n t = \bar{x}$ et il suit, à cause de la continuité de f et T , que $f\bar{x} = \bar{x} \in T\bar{x}$.

Observation. Le Théorème 1 est vrai aussi si on remplace la condition (iii) par

l'une des inégalités suivantes:

$$H^2(Tx, Ty) < \frac{D^2(fx, Tx)D^2(fx, Ty)}{D(fx, Tx)D(fy, Ty) + D(fx, Ty)D(fy, Tx)},$$

$$H^2(Tx, Ty) < D^n(fx, Tx)D(fy, Ty) + aD^n(fx, Ty)D(fy, Tx),$$

$$H^2(Tx, Ty) < D^n(fx, Tx)D(fy, Ty) + aD^n(fx, Ty)D^n(fy, Tx),$$

pour tous $x, y \in X$, $x \neq y$ et $a \in \mathbb{R}_+$;

$$H^2(Tx, Ty) < a(D(fx, Tx)D(fy, Ty) + D(fx, Ty)D(fy, Tx)) +$$

$$+ b(D(fx, Tx)D(fy, Tx) + D(fx, Ty)D(fy, Ty)) +$$

$$+ (c(D(fx, Tx)D(fy, Tx) + D(fx, Ty)D(fy, Ty)))$$

pour tous $x, y \in X$, $x \neq y$ et $a, b, c \geq 0$, tels que $a + 2b + 2c < 1$.

Théorème 2. Soient (X, d) un espace métrique complet, $f : X \rightarrow X$ et $T : X \rightarrow K(X)$ des applications continues. Si on a:

$$(i) \quad T(X) \subseteq f(X),$$

$$(ii) \quad T \text{ et } f \text{ sont faiblement commutatives,}$$

$$(iii) \quad H^2(Tx, Ty) \leq h \max \{d^2(fx, fy) D(fx, Tx)D(fy, Ty), D(fx, Ty)D(fy, Tx)\} \text{ pour tous } x, y \in X, x \neq y, 0 \leq h < 1,$$

alors f et T ont un point de coïncidence $t \in X$ ($ft \in Tt$).

De plus, si T et f satisfont à l'une des conditions (iv) du Théorème 1, alors T et f ont un point fixe commun $t \in X$.

Démonstration. Soit $x_0 \in X$ un point arbitraire et on choisit $x_1 \in X$ tel que $fx_1 \in Tx_0$ (on peut le faire car $Tx_0 \subseteq f(X)$).

Si $h = 0$ alors $D^2(fx_1, Tx_1) \leq H^2(Tx_0, Tx_1) = 0$, c'est-à-dire $fx_1 \in Tx_1$ parce que Tx_1 est un ensemble fermé. On suppose que $h > 0$ et alors, de la définition de H , il existe un point $y_1 \in Tx_1$ tel que $d(y_1, fx_1) \leq H(Tx_1, Tx_0)$. Car $Tx_1 \subseteq f(X)$, soit $x_2 \in X$ tel que $y_1 = fx_2$.

En général, on choisit $x_{n+1} \in X$ tel que $y_n = fx_{n+1} \in Tx_n$ et $d(y_n, fx_n) \leq H(Tx_n, Tx_{n-1})$ pour tout $n \geq 1$. En employant la relation (iii) du Théorème 2 on a

$$d^2(fx_n, fx_{n+1}) \leq H^2(Tx_{n-1}, Tx_n) \leq h \max \{d^2(fx_{n-1}, fx_n),$$

$$D(fx_{n-1}, Tx_{n-1})D(fx_n, Tx_n), D(fx_{n-1}, Tx_n)D(fx_n, Tx_{n-1})\}$$

et donc

$$d^2(fx_n, fx_{n+1}) \leq h \max \{d^2(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n)d(fx_n, fx_{n+1})\},$$

ce que est vrai parce que $fx_n \in Tx_{n-1}$ et Tx_{n-1}, Tx_n sont des ensembles proximinales. Il y a deux cas:

a) $\max \{d^2(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n)d(fx_n, fx_{n+1})\} = d^2(fx_{n-1}, fx_n)$ et alors on a

$$(1) \quad d(fx_n, fx_{n+1}) \leq \sqrt{h}d(fx_{n-1}, fx_n), \quad \forall n \in \mathbb{N};$$

b) $\max \{d^2(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n)d(fx_n, fx_{n+1})\} = d(fx_{n-1}, fx_n)d(fx_n, fx_{n+1})$ et alors on a

$$(2) \quad d(fx_n, fx_{n+1}) \leq hd(fx_{n-1}, fx_n), \quad \forall n \in \mathbb{N}.$$

Mais les relations (1) ou (2) montrent que la suite $\{fx_n\}$ est une suite de Cauchy et, car l'espace métrique (X, d) est complet, il suit qu'il existe $t \in X$ tel que $fx_n \rightarrow t$ pour $n \rightarrow \infty$.

Le fait que $ft \in Tt$ et l'affirmation (iv) se démontrent d'une manière analogue de celle de la démonstration de notre premier théorème.

Observations. 1. Le Théorème 2 est vrai si on remplace l'inégalité (iii) par l'inégalité suivante:

$$H^2(Tx, Ty) < \max \{D(fx, Tx)D(fy, Ty), D(fx, Ty)D(fy, Tx)\},$$

$$\forall x, y \in X, \quad x \neq y.$$

2. Les Théorèmes 1, 2 et leurs conséquences sont vraies aussi dans la situation quand $f, T : X \rightarrow X$ sont des autoapplications continues de l'espace métrique complet (X, d) .

3. Si $f : X \rightarrow X$ est l'application identique, on obtient des Théorèmes 1, 2 et de leurs conséquences des résultats ressemblants de ceux de Nicoleta Negoescu [6], [7] pour une seule application multivoque.

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BIBLIOGRAPHIE

1. Engelking R., *Outline of General Topology*. Amstredam - Warszawa, 1968.
2. Jungck G., *Commuting Mappings and Fixed Points*. Amer. Math. Monthly, **83**, 261-263 (1976).
3. Jain R. K., Dixit S. P., *Some Fixed Point Theorems for Mappings in Pseudocompact Tihonov Space*. Publ. Math. Debrecen, **33**, 195-197 (1986).
4. Kaneko H., *A General Principle for Fixed Points of Contractive-Multivalued Mappings*. Math. Japon., **31**, (3), 407-411 (1986).
5. Kaneko H., *A Common Fixed Point of Weakly Commuting Multi-valued Mappings*. Math. Japon., **33**, (5), 741-744 (1988).
6. Negoescu Nicoleta, *Observations sur des paires d'applications multivoques d'un certain type de contractivité*. Bul. Inst. Polit. Iași, **XXXV (XXXIX)**, 3-4, s. I, 21-25 (1989).

7. Negoescu Nicoleta, *Points fixes communs pour des applications multivoques qui satisfont aux inégalités rationnelles*. Bul. Inst. Polit. Iași, **XLII (XLVI)**, 1-2, s. I, 29-34 (1996).
8. Rao K. P. R., *Coincidence Point for Maps of Jungk Type*. Math. Japonica, **32**, 89-93 (1987).
9. Sessa S., *On a Weak Commutativity Condition of Mappings in Fixed Point Considerations*. Publ. Inst. Math. (Beograd), **32 (46)**, 146-153 (1982).

PUNCTE FIXE COMUNE ȘI PUNCTE DE COINCIDENȚĂ PENTRU OPERATORI ȘI MULTIFUNCȚII SLAB COMUTATIVE

(Rezumat)

Se demonstrează două teoreme de punct fix comun pentru un operator $f : X \rightarrow X$ și o multifuncție $T : X \rightarrow K(X)$ ((X, d) spațiu metric complet) slab comutativi care satisfac condițiile contractive (iii) din aceste teoreme, plecând de la rezultatele obținute de către S. Sessa [9], G. Jungck [2], H. Kaneko [5], R. K. Jain & S. P. Dixit [3] și Nicoleta Negoescu [6], [7].

INTEGRALE ANALYTIQUE DES MULTI-APPLICATIONS

PAR

AL. SABADIȘ

Abstract. The notion of analytic integral of a multifunction is introduced, starting from the Ne y's concept of boolean integral.

Key words: analytic convergence, multifunction, analytic integrability.

1. Notions et résultats concernant l'intégrale booléenne

Définition 1.1. Soit X un ensemble infini. Une famille de parties de X , $\{X_1, X_2, \dots, X_n\}$, s'appelle partition finie de X si on a

$$X = \bigcup_{i=1}^n X_i, \text{ ou } X_i \neq \emptyset, i = \overline{1, n}, \text{ et } X_i \cap X_j = \emptyset,$$

si $i \neq j$, $i = \overline{1, n}$, et $j = \overline{1, n}$. On la note par $d(X) = \{X_1, X_2, \dots, X_n\}$ et les parties X_i , $i = \overline{1, n}$, s'appellent parcelles.

Définition 1.2. Une partition $d(X)$ sépare deux éléments de X , si les deux éléments sont inclus dans deux parcelles distinctes.

Définition 1.3. Une partition $d''(X)$ est plus fine que $d'(X)$, si $d''(X)$ sépare tous les couples d'éléments de X que $d'(X)$ les sépare, et de plus, $d''(X)$ sépare au moins un couple d'éléments qui ne sont pas séparés par $d'(X)$. On note, $d'(X) < d''(X)$.

Définition 1.4. Une famille infini de partitions de X , $\{d_\alpha(X)\}_{\alpha \in I} = \mathcal{D}$, s'appelle infiniment raffinante si on a:

a) $\mathcal{D}$ est dirigée concernant la finesse, dans le sens que pour tout $d', d'' \in \mathcal{D}$, il existe $d^* \in \mathcal{D}$ tel que $d' < d^*$ et $d'' < d^*$.

b) Pour tout $x, y \in X$, $x \neq y$, il existe une partition $d \in \mathcal{D}$ qui les sépare.

On dit que $\mathcal{D}$ détermine un processus de raffinement infini des partitions de X , et on la note par $d_\alpha(X) \rightarrow d_\infty(X)$.

L'ensemble X , muni d'une famille infiniment raffinante des partition s'appelle ensemble principal; $\mathcal{D}$ s'appelle parcelllement principal.

Définition 1.5. Soient X et Y deux ensembles quelconques et $d(X) = \{X_1, X_2, \dots, X_n\}$ une partition de X . Dans X on met en évidence le parcelllement fini d , ayant n parcelles, tandis que dans Y on met en évidence tout au plus n sous-ensembles, d'après une loi notée par σ . On dira que X est d -parcellé et Y est σ -marqué; on notera $X = \bigcup_{i=1}^n X_i$, $X_i \cap X_j = \emptyset$ si $i \neq j$, $i = \overline{1, n}$ et $j = \overline{1, n}$, et par $\sigma = \{Y_1, Y_2, \dots, Y_m\}$, $m < n$, les σ -parties de Y mis en correspondance avec X_k , en mentionnant qu'on puisse avoir $Y_i = Y_j$, même si $i \neq j$ (dans le cas où $m < n$). Donc, $X_k \rightarrow Y_k$, $k = \overline{1, n}$.

La réunion $S_\sigma(d) = \bigcup_{k=1}^n Y_k \times X_k$, s'appelle réunion intégrale.

Définition 1.6. Soit $S_\sigma(d)$, la famille des réunions intégrales correspondantes à une famille infiniment raffinante de partitions de X . On dit qu'un ensemble $I \subseteq Y \times X$ est la limite analytique de la suite généralisée $(S_\sigma(d))$, ($I = (a) - \lim_{d \rightarrow d_\infty} \bigcup_d (Y_k \times X_k)$), s'il existe une suite d'ensembles $(B_p) \subset P(Y \times X)$, $B_p \downarrow \emptyset$, et une famille infiniment raffinante de partitions de X , $\mathcal{D}$, duquelle on peut extraire une suite infiniment raffinante de partitions (d_{B_p}) , $p \in \mathbb{N}$, telle que pour tout $d \in \mathcal{D}$, avec $d_{B_p} \prec d$, on a

$$\left(\bigcup_d (Y_k \times X_k) \right) \Delta I \subseteq B_p.$$

L'ensemble $I \subseteq Y \times X$ s'appelle l'intégrale booléenne de l'ensemble Y par rapport à l'ensemble X , si le passage à la limite est indiqué par $d \rightarrow d_\infty$.

Définition 1.7. Soient $\mathcal{D}$ une famille infiniment raffinante de partitions de l'ensemble X , et Y un ensemble quelconque. La suite généralisée $(A_d)_d \subset P(Y \times X)$, s'appelle "suite (a) -Cauchy généralisée" s'il existent une suite $(B_p)_{p \in \mathbb{N}} \subset P(Y \times X)$, $B_p \downarrow \emptyset$, et une suite infiniment raffinante de partitions, $(d_{B_p}) \subset \mathcal{D}$, telles que pour tout $d', d'' \in \mathcal{D}$ qui satisfont les relations $d_{B_p} \prec d'$ et $d_{B_p} \prec d''$, a lieu $A_{d'} \Delta A_{d''} \subseteq B_p$.

Théorème 1.1. Soient X et Y deux ensembles quelconques dans lesquels on met en évidence des sous-ensembles d'après les lois π , respectivement, σ . On suppose que π joue le rôle du parcelllement principal (l'on note par d), faisant part d'une famille infiniment raffinante de parcelllements, $\mathcal{D}$ (π et σ seront coordonnés qu'on puisse appliquer l'ensemble des parcelles de l'ensemble principal sur l'ensemble des parties mises en évidence sur l'autre ensemble).

Il existe la limite analytique, dans le sens de la Définition 1.6, $I = (a) - \lim_{d \rightarrow d_\infty} \bigcup_d Y_k \times X_k \subseteq Y \times X$, si et seulement si la suite généralisée $\left(\bigcup_d Y_k \times X_k \right)_{d \in \mathcal{D}}$ est une "suite (a) -Cauchy généralisée" (v. Théorème 4 [1]).

Remarque. " σ " pourra, aussi, jouer le rôle du parcelllement principal. Dans ce

cas les réunions intégrales auront la forme $\left(\bigcup_d X_k \times Y_k\right)_{d \in \mathcal{D}}$

Théorème 1.2. Soient $\mathcal{C}' \subset P(X)$ et $\mathcal{C}'' \subset P(Y)$, des tribes, et $m' : \mathcal{C}' \rightarrow \mathbb{R}$, respectivement, $m'' : \mathcal{C}'' \rightarrow \mathbb{R}$, des mesures complètes. On note par m , la mesure induite par m' et m'' sur le tribe $\mathcal{C} \subset P(Y \times X)$, généré par la famille des réunions finies et disjoints d'ensemble de la famille, $F = \{A \times B \mid A \in \mathcal{C}' \text{ et } B \in \mathcal{C}''\} \subset P(Y \times X)$. Si l'ensemble X est muni d'une famille infiniment raffinant de partitions $\mathcal{D}$ et si la suite généralisée des réunions intégrales $\left(\bigcup_d Y_k \times X_k\right)_{d \in \mathcal{D}}$, $Y_k \in \mathcal{C}''$, $X_k \in \mathcal{C}'$ est une suite

(a)- Cauchy généralisée en (R, ρ_u) , et il existe la limite finie, $\lim_{d \rightarrow d_\infty} m \left(\bigcup_d Y_k \times X_k\right) < +\infty$.

Théorème 1.3. S'il existe l'intégrale booléenne I de l'ensemble Y par rapport à l'ensemble X , et si les hypothèses du Théorème 1.2 sont vérifiées, alors a lieu

$$\lim_{d \rightarrow d_\infty} m \left(\bigcup_d Y_k \times X_k\right) = m \left((a) - \lim_{d \rightarrow d_\infty} \bigcup_d Y_k \times X_k\right) < +\infty.$$

2. La notion d'intégrale analytique d'une multi-application

On considère deux ensembles quelconques, X et Y , et on note par (X, Y, F) une multi-application surjective définie sur X aux valeurs dans Y . Dans ces conditions, à chaque partition finie de X , $d = \{X_1, X_2, \dots, X_n\}$, correspond un F -marquage de Y , $\{Y_1, Y_2, \dots, Y_n\}$, où $Y_i = \bigcup_{x \in X_i} F(x)$, $i = \overline{1, n}$.

On note par $\mathcal{D}$ une famille infiniment raffinant de partitions de X . La réunion $S(d, F) = \bigcup_d F(X_k) \times Y_k$ s'appelle réunion intégrale de la multi-application F , correspondante à la partition $d \in \mathcal{D}$.

Définition 2.1. Dans les conditions présentées plus haut, on dit que la multi-application (X, Y, F) est (a)-intégrable sur X , s'il existe $I \subseteq Y \times X$ tel que $(a) - \lim_{d \rightarrow d_\infty} S(d, F) = I$. L'ensemble I s'appelle l'intégrale analytique de F , sur X .

Observation 2.1. Si, de plus, (X, Y, F) est une multi-application injective, c'est-à-dire de $x' \neq x''$ il résulte $F(x') \cap F(x'') = \emptyset$, alors le F -marquage de Y est une partition. Donc dans ce cas on peut prendre Y comme ensemble principal.

Définition 2.2. Soit $d = \{Y_1, Y_2, \dots, Y_n\}$ une partition appartenant à une famille infiniment raffinant de partitions de l'ensemble Y . Dans ces conditions, la famille d'ensembles $\{F^-(Y_1), F^-(Y_2), \dots, F^-(Y_n)\}$ constitue un marquage de l'ensemble X . L'ensemble $I^- \subseteq X \times Y$ s'appelle intégrale analytique de l'ensemble Y par rapport à l'ensemble X , réalisée par l'application $(P(Y), P(X), F^-)$, si on a

$$(a) - \lim_{d \rightarrow d_\infty} \bigcup_d F^-(Y_k) \times Y_k = I^-.$$

Théorème 2.1. Soient X et Y deux ensembles quelconques, et (X, Y, F) une multi-application surjective. On suppose que X est muni d'une famille infiniment raffinant de partitions $\mathcal{D}$. F est (a) -intégrable sur X , si et seulement si la suite généralisée $\left(\bigcup_d F(X_k) \times X_k\right)_{d \in \mathcal{D}}$ et une suite (a) -Cauchy généralisée, c'est-à-dire il existe une suite $(D_p) \subset P(Y \times X)$, $D_p \downarrow \emptyset$, et une suite infiniment raffinant de partitions attachée à (D_p) , $(d_{D_p}) \subset \mathcal{D}$, telles que pour toutes $d', d'' \in \mathcal{D}$ avec $d_{D_p} \prec d'$ et $d_{D_p} \prec d''$, on a

$$(*) \quad \left[\bigcup_{d'} F(X'_k) \times X'_k \right] \Delta \left[\bigcup_{d''} F(X''_k) \times X''_k \right] \subseteq D_p.$$

Démonstration. Si on a $I = (a) - \lim_{d \rightarrow d_\infty} \bigcup_d F(X_k) \times X_k$, $I \subseteq Y \times X$, alors il existe une suite $(D_p) \subset Y \times X$, $D_p \downarrow \emptyset$, telle qu'on a

$$\left(\bigcup_{d'} F(X'_k) \times X'_k \right) \Delta I \subseteq D_p \quad \text{si} \quad d_{D_p} \prec d',$$

$$\left(\bigcup_{d''} F(X''_k) \times X''_k \right) \Delta I \subseteq D_p \quad \text{si} \quad d_{D_p} \prec d''.$$

Donc, on a

$$\begin{aligned} & \left(\bigcup_{d'} F(X'_k) \times X'_k \right) \Delta \left(\bigcup_{d''} F(X''_k) \times X''_k \right) \subseteq \\ & \subseteq \left[\left(\bigcup_{d'} F(X'_k) \times X'_k \right) \Delta I \right] \cup \left[\left(\bigcup_{d''} F(X''_k) \times X''_k \right) \Delta I \right] \subseteq D_p \end{aligned}$$

si $d_{D_p} \prec d'$ et $d_{D_p} \prec d''$, et la nécessité est démontrée.

Réciproquement, on choisit de la famille $\mathcal{D}$ une suite de partitions strictement monotone et infiniment raffinant, $d_1 \prec d_2 \prec \dots \prec d_m \prec \dots$, $d_m \rightarrow d_\infty$ et on construit une suite de réunions intégrales correspondante à la suite (d_m) , notamment $\left(\bigcup_{d_m} F(X_k) \times X_k\right)_{m \in \mathbb{N}}$. Vu la condition $(*)$, cette suite est une suite (a) -Cauchy généralisée et par suite elle a une limite dans $P(Y \times X)$. Soit

$$I = (a) - \lim_{d_m \rightarrow d_\infty} \bigcup_{d_m} F(X_k) \times X_k.$$

Pour une partition arbitraire $d \in \mathcal{D}$, on a

$$\begin{aligned} & \left(\bigcup_d F(X_i) \times X_i \right) \Delta I \subseteq \left[\left(\bigcup_d F(X_i) \times X_i \right) \Delta \left(\bigcup_{d_m} F(X_k) \times X_k \right) \right] \cup \\ & \cup \left[\left(\bigcup_{d_m} F(X_k) \times X_k \right) \Delta I \right] \subseteq D_p \cup D'_p = D''_p, \end{aligned}$$

si $d, d_m \succ d_{D_p}$ et $d_m \succ d_{D_p'}$.

Donc, on a $\left(\bigcup_d F(X_i) \times X_i\right) \Delta I \subseteq D_p'' \downarrow \emptyset$, si $d \succ d_{D_p''}$, où $d_{D_p''} \succ d_{D_p}, d_{D_p'}$. Le théorème est démontré.

T h é o r è m e 2.2. Soient $\mathcal{C}' \subset P(X)$ et $\mathcal{C}'' \subset P(Y)$ des tribes $m' : \mathcal{C}' \rightarrow [0, \infty)$ et $m'' : \mathcal{C}'' \rightarrow [0, \infty)$ des mesures complètes et finies sur X , respectivement sur Y . On note par m la mesure induite de m' et m'' sur $Y \times X$, $m : \mathcal{C}_{Y \times X} \rightarrow [0, \infty)$. On indique, aussi par une famille infiniment raffinant de partitions du X et par (X, Y, F) une multi-application surjective

Si la suite généralisée $\left(\bigcup_d F(X_k) \times X_k\right)_{d \in \mathcal{D}} \subset \mathcal{C}_{Y \times X}$ est une suite (a)-Cauchy généralisée, alors la suite numérique $m \left(\bigcup_d F(X_k) \times X_k\right)_{d \in \mathcal{D}}$ est une suite de Cauchy généralisée et elle a une limite finie, $\lim_{d \rightarrow d_\infty} m \left(\bigcup_d F(X_k) \times X_k\right)$.

D é m o n s t r a t i o n. Puisque $\left(\bigcup_d F(X_k) \times X_k\right)_{d \in \mathcal{D}}$ est une suite (a)-Cauchy généralisée, pour $d', d'' \succ d_{D_p}$ on a

$$\left[\left(\bigcup_{d'} F(X'_i) \times X'_i \right) - \left(\bigcup_{d''} F(X''_j) \times X''_j \right) \right] \cup \\ \cup \left[\left(\bigcup_{d''} F(X''_j) \times X''_j \right) - \left(\bigcup_{d'} F(X'_i) \times X'_i \right) \right] \subseteq D_p \downarrow \emptyset$$

(voir la relation (*)). Donc, ont lieu:

- 1) $\left(\bigcup_{d'} F(X'_i) \times X'_i \right) - \left(\bigcup_{d''} F(X''_j) \times X''_j \right) \subseteq D_p \Rightarrow m \left(\bigcup_{d'} \right) - m \left(\bigcup_{d''} \right) \leq m(D_p);$
- 2) $\left(\bigcup_{d''} F(X''_j) \times X''_j \right) - \left(\bigcup_{d'} F(X'_i) \times X'_i \right) \subseteq D_p \Rightarrow m \left(\bigcup_{d''} \right) - m \left(\bigcup_{d'} \right) \leq m(D_p).$

De 1) et 2) on a

$$\left| m \left(\bigcup_{d'} F(X'_i) \times X'_i \right) - m \left(\bigcup_{d''} F(X''_j) \times X''_j \right) \right| \leq m(D_p).$$

Vu que $m(D_p) \xrightarrow[p \rightarrow \infty]{} 0$, le théorème est démontré.

C o n s é q u e n c e 1. Si, de plus, la multi-application F est (a)-intégrable sur X , alors a lieu

$$\lim_{d \rightarrow d_\infty} m \left(\bigcup_d F(X_k) \times X_k \right) = m \left((a) - \lim_{d \rightarrow d_\infty} \bigcup_d F(X_k) \times X_k \right).$$

D é m o n s t r a t i o n. Par hypothèse on a $(a) - \lim_{d \rightarrow d_\infty} \bigcup_d F(X_k) \times X_k = I$, $I \subseteq Y \times X$, c'est-à-dire il existe $(D_p) \subset P(Y \times X)$, $D_p \downarrow \emptyset$, telle qu'on a $\left(\bigcup_d F(X_k) \times X_k\right) \Delta I \subseteq D_p$ dès que $d \succ d_{D_p}$.

En procédant comme dans la démonstration du Théorème 2.2 on obtient $m\left(\bigcup_d F(X_k) \times X_k\right) - m(I) \leq m(D_p)$, d'où que $\lim_{d \rightarrow d_\infty} m\left(\bigcup_d F(X_k) \times X_k\right) = m(I)$, ce qu'il fallait démontrer.

C o n s é q u e n c e 2. Si tout $X_k \in d \in \mathcal{D}$, $k = \overline{1, n}$, $X_k \in C'$ et $F(X_k) \in C''$, alors on a

$$\lim_{d \rightarrow d_\infty} \bigcup_d m''(F(X_k))m'(X_k) = m\left((a) - \lim_{d \rightarrow d_\infty} \bigcup_d F(X_k) \times X_k\right)$$

La démonstration est immédiate, vu que

$$m\left(\bigcup_d F(X_k) \times X_k\right) = \sum_d m''(F(X_k))m'(X_k).$$

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BIBLIOGRAPHIE

1. N e y A., *Contributions à la théorie de la convergence des suites et des séries d'ensembles* (II). *Mathematica*, 14 (37), 2 (1972).

INTEGRALA ANALITICĂ A MULTIFUNCȚIILOR

(Rezumat)

Se introduce noțiunea de multifuncție integrabilă în sens analitic și se dă o condiție necesară și suficientă de integrabilitate analitică.

CONSIDERATION ON SOLVING SOME DIOPHANTINE EQUATIONS WITH APPLICATIONS IN TECHNICS (MATRIX SOLUTIONS)

BY

M. ISPAS

Abstract. The papers studies the solutions to some diophantine equations, including one which generalizes Pell's equation.

Key words: diophantine equation, Pell equation, matrix solution.

A. Definition 1. A first degree *diophantine equation* in two unknowns is an equation of the form

$$(1) \quad ax + by = c,$$

where a, b, c are integer numbers and the pair of unknowns (x, y) is in $\mathbb{Z} \times \mathbb{Z}$.

Remark. Eq. (1) is called a diophantine equation not because $a, b, c \in \mathbb{Z}$ but since the unknown pair (x, y) is in $\mathbb{Z} \times \mathbb{Z}$.

Let d = the greatest common divisor of $\{a, b\}$, denoted by $\langle a, b \rangle$.

Proposition 1. *The necessary and sufficient condition for the diophantine equation (1) to admit solutions is that $d|c$ (d divides c). If (x_0, y_0) is a particular solution of Eq. (1), then its general solution is*

$$\begin{cases} x = x_0 + tv; & v = b : d \\ y = y_0 - tu; & u = a : d \end{cases} \quad \text{with} \quad t \in \mathbb{R} \text{ (arbitrary).}$$

Proof.

$$\begin{cases} ax + by = c, \\ ax_0 + by_0 = c, \end{cases} \Rightarrow a(x - x_0) + b(y - y_0) = 0$$

and therefore we can take $x - x_0 = (bt)/d$ and $y - y_0 = -(at)/d$ (with $t \in \mathbb{R}$) for the equation to be satisfied. We first solve the homogeneous diophantine equation of degree one in three unknowns x, y, z

$$(2) \quad ax + by + cz = 0$$

and then – as a particular case – Eq. (1).

Let the determinant

$$\begin{vmatrix} a & b & c \\ a & b & c \\ m & n & p \end{vmatrix} = 0 \quad (a, b, c, m, n, p \in \mathbb{Z}) \Rightarrow a(bp - cn) + b(cm - ap) + c(an - bm) = 0,$$

and therefore

$$\begin{cases} x = bp - cn, \\ y = cm - ap, \\ z = an - bm \end{cases} \quad \text{with arbitrary } m, n, p \in \mathbb{Z},$$

is the general solution of Eq. (2).

Example. Solve the diophantine equation $29x - 42y = 5$. Since the largest common divider $d = \langle 29, 42 \rangle = 1$ and $1/5 \Rightarrow$ the equation has solutions.

Let us consider the homogeneous diophantine equation $29x - 42y - 5z = 0$. From

$$\begin{vmatrix} 29 & -42 & -5 \\ 29 & -42 & -5 \\ m & n & p \end{vmatrix} = 0 \Rightarrow 29(-42p + 5n) - 42(-29p - 5m) - 5(29n + 42m) = 0 \Rightarrow$$

$$\Rightarrow \begin{cases} x = -42p + 5n, \\ y = -29p - 5m, \\ z = 29n + 42m, \end{cases} \quad m, n, p \in \mathbb{Z}.$$

Using condition $z = 1 = 29n + 42m$ we find $n_0 = -13$, $m_0 = 9$. It follows that $x = -42p - 65$, $y = -29p - 45$, with $p \in \mathbb{Z}$ is the general solution of the given equation.

B. Proposition 2. Let $x, y \in \mathbb{N}^*$ and $\langle x, y \rangle = 1$ (x and y are mutually prime). Then $\langle x + y, xy \rangle = 1$, too.

Proof. Let $p \geq 2$ be a common prime divisor for $x + y$ and xy , what implies $p|(x + y)$ and $p|xy$; from the latter division relationship we have $\left. \begin{matrix} (x, y) = 1 \\ p|xy \end{matrix} \right\} \Rightarrow p/x$ and $p \nmid y$, or $p \nmid x$ and p/y . But $p|(x + y)$ what is a contradiction.

Proposition 3. The solutions in natural (positive integer) numbers of the equation

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2} \text{ are } \begin{cases} x = t(u^2 + v^2)(u^2 - v^2), \\ y = t(u^2 + v^2)2uv, \\ z = t(u^2 + v^2)uv, \end{cases} \quad u, v \in \mathbb{N}^*, \langle u, v \rangle = 1, u > v, t \in \mathbb{N}^*.$$

Proof. This equation is equivalent to $z^2(x^2 + y^2) = x^2y^2$. Let $d = \langle x, y \rangle \Rightarrow x = dx_1$, $y = dy_1$, where $\langle x_1, y_1 \rangle = 1$. Therefore

$$(3) \quad z^2(x_1^2 + y_1^2) = d^2x_1^2y_1^2.$$

Since $\langle x_1, y_1 \rangle = 1$ it follows that $\langle x_1^2, y_1^2 \rangle = 1 \Rightarrow \langle x_1^2 + y_1^2, x_1^2 y_1^2 \rangle = 1$. From (3) $\Rightarrow z^2 = kx_1^2 y_1^2$ and $k(x_1^2 + y_1^2) = d^2$, $k \in \mathbb{N}^*$, which is a perfect square (if we take $k = t^2$, $t \in \mathbb{N}^*$). Then $z = tx_1 y_1$ and $t^2(x_1^2 + y_1^2) = d^2 \Rightarrow x_1^2 + y_1^2 = l^2$ ($l = d/t$). We thus obtain a Pythagoreic equation whose general solutions are known:

$$\begin{cases} x_1 = A(u^2 - v^2), \\ y_1 = A2uv, \\ l = A(u^2 + v^2), \end{cases} \quad \langle u, v \rangle = 1, \quad u > v, \quad A \in \mathbb{N}^* \quad (\text{arbitrary}).$$

From $\langle x_1, y_1 \rangle = 1 \Rightarrow A = 1$ and we finally obtain the solution given in the statement.

Remark. In the particular case when $t = 1$, $u = \frac{1}{2}(n+1)$, $v = \frac{1}{2}(n-1)$ with $n \in \mathbb{N}^*$, we find the integer solutions of equation $1/x^2 + 1/y^2 = 1/z^2$ as given by $x = \pm n \frac{1}{2}(1+n^2)$, $y = \pm \frac{1}{2}(n^2-1) \frac{1}{2}(n^2+1)$, $z = \pm n \frac{1}{2}(n^2-1)$, $n = 2k+1$, $k \in \mathbb{N}$.

C. Equation

$$(4) \quad x^2 + y^2 = z^2$$

admits the solutions

$$\begin{cases} x = A^2 - B^2 & (\text{or } B^2 - A^2), \\ y = 2AB, \\ z = A^2 + B^2, \end{cases}$$

even in the case when A, B are (unspecified) mathematical objects in a structure endowed with addition and multiplication operations (the latter being commutative).

Examples.

$$1. \quad A = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 15 & 2 \end{pmatrix}, \quad \text{for which } AB = BA = \begin{pmatrix} 50 & 28 \\ 70 & 36 \end{pmatrix} \Rightarrow$$

$$\Rightarrow X = B^2 - A^2 = \begin{pmatrix} 89 & 28 \\ 70 & 75 \end{pmatrix}, \quad Y = 2AB = \begin{pmatrix} 100 & 56 \\ 140 & 72 \end{pmatrix}, \quad Z = A^2 + B^2 = \begin{pmatrix} 141 & 56 \\ 190 & 113 \end{pmatrix}.$$

$$A = \begin{pmatrix} -1 & 1 & 2 \\ -3 & 2 & 1 \\ 5 & 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 & 4 \\ -6 & 5 & 2 \\ 10 & 8 & 7 \end{pmatrix}, \quad \text{for which } AB = BA = \begin{pmatrix} 15 & 19 & 12 \\ 1 & 12 & -1 \\ 1 & 54 & 49 \end{pmatrix} \Rightarrow$$

$$\Rightarrow X = B^2 - A^2 = \begin{pmatrix} 21 & 31 & 23 \\ -6 & 24 & 1 \\ 14 & 91 & 82 \end{pmatrix}, \quad Y = 2AB = \begin{pmatrix} 30 & 28 & 24 \\ 2 & 24 & -2 \\ 2 & 108 & 98 \end{pmatrix}, \quad Z = A^2 + B^2 = \begin{pmatrix} 37 & 49 & 33 \\ -2 & 34 & -1 \\ 10 & 141 & 128 \end{pmatrix}.$$

3. Let us now consider the equation $X^2 + Y^2 + Z^2 = T^2$, which admits the solutions

$$\begin{cases} X = A^2 + B^2 - C^2, \\ Y = 2AC, \\ Z = 2BC, \\ T = A^2 + B^2 + C^2, \end{cases}$$

where A, B, C are square matrices that mutually commute (their products are commutative).

For

$$A = \begin{pmatrix} 4 & 5 \\ -3 & 1 \end{pmatrix}, B = \begin{pmatrix} 3 & 5 \\ -3 & 0 \end{pmatrix}, C = \begin{pmatrix} 2 & 5 \\ -3 & -1 \end{pmatrix}$$

we obtain

$$X = \begin{pmatrix} 6 & 35 \\ -21 & -15 \end{pmatrix}, Y = \begin{pmatrix} -14 & 30 \\ -18 & -32 \end{pmatrix}, Z = \begin{pmatrix} -18 & 20 \\ -12 & -30 \end{pmatrix}, T = \begin{pmatrix} -16 & 45 \\ -27 & -43 \end{pmatrix}.$$

D. We are now going to solve the equation $ax^2 - by^2 = 1$ in natural (positive integer) numbers, where a, b are given natural numbers.

Remark. This equation is more general than Pell's equation $u^2 - dv^2 = 1$, $d \in \mathbb{N}^*$.

Proposition 4. If $d \in \mathbb{N}^*$ is not a perfect square, then $u^2 - d^2v^2 = 1$ has infinitely many solutions in natural numbers and its general solution is (u_n, v_n) , $n \in \mathbb{N}^*$, where

$$\begin{cases} u_{n+1} = u_0u_n + dv_0v_n, \\ v_{n+1} = v_0u_n + u_0v_n \end{cases}, \quad u_1 = u_0, \quad v_1 = v_0,$$

(u_0, v_0) being the least (first) solution in natural numbers of this equation.

Proof. $(1, 0)$ is clearly a solution of Pell's equation. Let (u_0, v_0) be another solution of its but $\neq (1, 0)$. Assume that (u_n, v_n) is also a solution of this equation. It follows that $u_{n+1}^2 - dv_{n+1}^2 = (u_0u_n + dv_0v_n)^2 - d(v_0u_n + u_0v_n)^2 = (u_0^2 - dv_0^2)(u_n^2 - dv_n^2) = 1 \Rightarrow (u_{n+1}, v_{n+1})$ is also a solution of Pell's equation.

Proposition 5. Let us consider the equation

$$(5) \quad ax^2 - by^2 = 1, \quad a, b \in \mathbb{N}^*;$$

if $ab = k^2$, $k \in \mathbb{N}^*$, then Eq. (5) has no solution in integer numbers.

Proof. If we assume that such a solution exists then $\Rightarrow \langle a, b \rangle = 1$ and from $ab = k^2 \Rightarrow a = k_1^2, b = k_2^2$, where $k_1k_2 = k$, $k_1, k_2 \in \mathbb{N}^*$. Eq. (5) becomes $k_1^2x^2 - k_2^2y^2 = 1 \Rightarrow 1 < k_1x + k_2y = k_1x - k_2y = 1$, which is false.

Definition 2. We say that a *resolvent* of Pell's Eq. (5) is

$$(6) \quad u^2 - abv^2 = 1.$$

Proposition 6. If equation $ax^2 - by^2 = 1$ has a nontrivial solution in natural (positive integer) numbers, then it has infinitely many such solutions.

Proof. Let $(x_0, y_0) \in \mathbb{N}^* \times \mathbb{N}^*$, be a solution of the equation in the statement. Since ab is not a perfect square, it follows from P_3 that Pell's resolvent has infinitely many solutions in natural numbers, obtained for $d = ab$.

Let (u_n, v_n) , $n \in \mathbb{N}$ be the general solution of Eq. (6). Then (x_n, y_n) , $n \in \mathbb{N}$, in which

$$\begin{cases} x_n = x_0u_n + by_0v_n, \\ y_n = y_0u_n + ax_0v_n, \end{cases}$$

are solutions of Eq. (5) for $(\forall)n \in \mathbb{N}$ since

$$ax_n^2 - by_n^2 = a(x_0u_n + by_0v_n)^2 - b(y_0u_0 + ax_0v_n)^2 = (ax_0^2 - by_0^2)(u_n^2 - av_n^2) = 1, \\ (\forall)n \in \mathbb{N}.$$

Proposition 7. The general solution of Eq. (5) is (x_n, y_n) , $n \in \mathbb{N}$ where

$$\begin{cases} x_n = Au_n + bBv_n, \\ y_n = Bu_n + aAv_n, \end{cases}$$

in which (u_n, v_n) , $n \in \mathbb{N}$ is the general solution to Pell's resolvent and (A, B) , $A, B \in \mathbb{N}$, is the least (first) solution of this equation.

Proof. Let (u_n, v_n) , $n \in \mathbb{N}$, be the general solution to Pell's resolvent. It follows from P₇ that (x_n, y_n) , $n \in \mathbb{N}$ are solutions to Eq. (5). Conversely,

Proposition 8. If (x_n, y_n) , $n \in \mathbb{N}$ are solutions to Eq. (5), then (u_n, v_n) , $n \in \mathbb{N}$, with

$$\begin{cases} u_n = aAx_n - bBy_n, \\ v_n = Bx_n - Ay_n, \end{cases}$$

are solutions to Pell's resolvent.

Indeed,

$$u_n^2 - av_n^2 = (aAx_n - bBy_n)^2 - ab(Bx_n - Ay_n)^2 = (aA^2 - bB^2)(ax_n^2 - by_n^2) = 1, \quad (\forall)n \in \mathbb{N}.$$

Remark. For using the matrix calculations, the general solutions in P₅ and P₈ can be written as

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} u_0 & dv_0 \\ v_0 & u_0 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \quad u_0 = u_1, \quad v_0 = v_1 \quad \text{and} \quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} A & bB \\ B & aA \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix}.$$

The second order matrices that occur above have determinants = 1 and therefore it is simpler to operate with such recurrences in this matrix format.

Example. Let us consider the equation $6x^2 - 5y^2 = 1$. The first solution is (1,1). Pell's resolvent is $u^2 - 30v^2 = 1$, with its first solution (11, 2). The general solution to Pell's resolvent is

$$\begin{cases} u_{n+1} = 11u_n + 60v_n, \\ v_{n+1} = 11v_n + 2u_n, \end{cases} \quad u_1 = 11, \quad v_1 = 2 \quad \text{and} \quad \begin{cases} x_n = u_n + 5v_n, \\ y_n = u_n + 6v_n, \end{cases}$$

being the general solution of the equation considered.

More generally, equation

$$(7) \quad \frac{1}{x} + \frac{1}{y} = \frac{1}{z},$$

has the general integer solution

$$\begin{cases} x = km(a+k), \\ y = am(a+k), \\ z = kma, \end{cases} \quad \text{with } m \in \mathbb{N}^*; \quad < k, a > = 1, \quad k, a \in \mathbb{Z}^*.$$

E. 1. Let us consider equation

$$(8) \quad x^2 + y^2 + z^2 = 3xyz, \quad x, y, z \in \mathbb{N}^*.$$

If (x, y, z) is a triple that satisfies this equation, then

$$(x, y, z) \xrightarrow{T_1} (3yz-x, y, z), \quad (x, y, z) \xrightarrow{T_2} (x, 3xz-y, z), \quad (x, y, z) \xrightarrow{T_3} (x, y, 3xy-z),$$

also verify it.

Any solution can be obtained starting from the initial solution $(1, 1, 1)$ and applying the transformations T_1, T_2 or T_3 finitely many times. We describe the generating process of the solutions by the directed graph presented in Fig. 1.

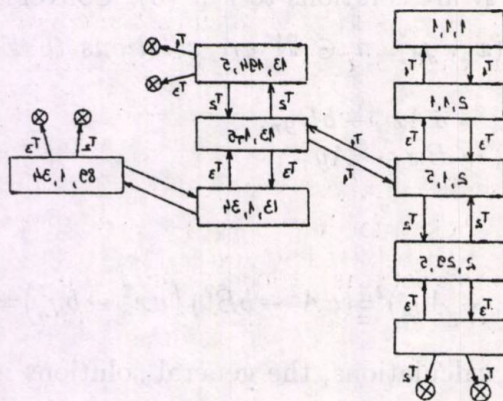


Fig. 1

2. Let us solve in $\mathbb{Z}$ diophantine equations of the form

$$(9) \quad (x + y)^{2p+1} = x^{2p} + y^{2p}, \quad \text{where } p \in \mathbb{N}^*, \quad p \geq 2.$$

Proposition 10. For any $p \in \mathbb{N}^*, p \geq 2$, we can write

$$(a + b)^{2p} = a^{2p} + b^{2p} + (a + b)^2 A(p) + a^p b^p B(p),$$

where $A(p)$ takes integer values, and $B(p) = (-1)^{p+1} 2$.

Proof. Starting from the Newton expansion of $(a + b)^n$ and taking $a = 1, b = -1, n = 2p$, we obtain

$$\begin{aligned} C_{2p}^1 - C_{2p}^2 + \dots + C_{2p}^{2p-1} = 2 &\Rightarrow 2C_{2p}^1 - 2C_{2p}^2 + \dots + (-1)^p 2C_{2p}^{p-1} + (-1)^{p+1} C_{2p}^p = 2 \Rightarrow \\ &\Rightarrow (-1)^p 2C_{2p}^1 + (-1)^{p-1} 2C_{2p}^2 + \dots + 2C_{2p}^{p-1} - C_{2p}^p = (-1)^p 2. \end{aligned}$$

Using this equation, the proof proceeds by mathematical induction. For Eq. (9) we exclude the trivial solution $(1, 0)$.

Let $d = \langle x, y \rangle$, $d > 0 \Rightarrow x = da$, $y = db$, with $\langle a, b \rangle = 1$. Eq. (9) becomes

$$d(a+b)^{2p+1} = a^{2p} + b^{2p}.$$

An analysis of the possible cases with respect to even/odd values of a , b implies that $(a+b)^{2p+1}$ and $a^{2p} + b^{2p}$ are mutually prime. Then $a+b=1 \Rightarrow b=1-a$ and $d = a^{2p} + (1-a)^{2p}$. We can write as

$$\begin{cases} x = a[a^{2p} + (1-a)^{2p}], \\ y = (1-a)[a^{2p} + (1-a)^{2p}], \end{cases} \quad a \in \mathbb{Z},$$

the general solution of Eq. (9).

3. Let us find the n -digit numbers ($n \in \mathbb{N}^*$), written in the decimal basis such that $\overline{a_1 a_2 \dots a_n} = a_1! + a_2! + \dots + a_n!$, where $\overline{a_1 a_2 \dots a_n}$ denotes just such a number.

For $n \geq 8$ we have $\overline{a_1 a_2 \dots a_n} \geq 10^{n-1}$ and $a_1! + a_2! + \dots + a_n! \leq n! = 362880n$ and – according to the given equality – we would have $10^{n-1} \leq 362880n$ (what is impossible for $n \geq 8$). Hence the possible numbers – solutions to this problem – have at most 7 digits and they should be looked for in the set $\{1, 2, 3, \dots, 9999999\}$.

To each number $N = \overline{a_1 a_2 \dots a_7}$ in this set it is associated the 7-tuple $(a_1, a_2, \dots, a_7)$. There obviously exists $s \in \mathbb{N}$ such that $a_1 = a_2 = \dots = a_s = 0$ and $a_{s+1} \neq 0$ and therefore N is a solution to the problem if

$$(10) \quad N = a_{s+1}! + \dots + a_7!.$$

The "filter" algorithm written in PASCAL checks each number in the set $\{1, \dots, 10^8 - 1\}$ and displays the ones that satisfy condition (10). This program finds as the only numbers satisfying this condition (10) are 1, 2, 145 and 40585.

Remark. Equations of the forms we have considered in this paper are met in problems of Material Strength, Mechanisms and Machine Details (the bases of splitting), Tool-Machines and other fields.

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REFERENCES

1. Sierpinski W., *Elementary Theory of Numbers*. Warszawa, 1964.
2. Ion I., Niță C., *Elemente de aritmetică cu aplicații în tehnică*. București, 1978.
3. Borevici Z. I., Safarevici I. R., *Teoria numerelor*. Ed. Științifică și Enciclopedică, București, 1985.
4. Korec I., *Diophantine Equations*. 1981.
5. Radovici-Mărculescu P., *Ecuatii diofantice fără soluții*. G.M.A., 1, 3 (1982).

CONSIDERAȚII PRIVIND REZOLVAREA UNOR ECUAȚII DIOFANTICE CU APLICĂȚII TEHNICE (SOLUȚII MATRICIALE)

(Rezumat)

Se cercetează soluțiile unor ecuații diofantice de forma (1)–(9), dintre care una mai generală decât cea a lui P e l l.

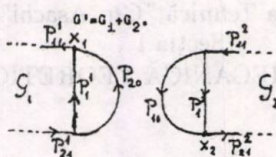


Fig. 2

Les équations de conservation sont évidemment satisfaites, parce que

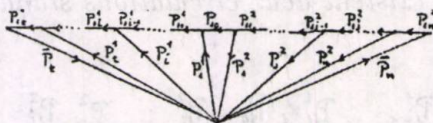
$$F_{10} = F_{20}.$$

Les fonctions de capacité c' et C' pour le graphe G' sont identiques à celui de G

$$c' = c \quad \text{et} \quad C' = C.$$

On vérifie que dans les graphes G_1 et G_2 il existent deux circulations standard admissibles de type $\bar{c}/\bar{b}'$, respectivement $\bar{b}'/\bar{c}$; la circulation dans le graphe G_1 est la restriction de la fonction de circulation admissible F dans le graphe G_1 , mais la circulation dans le graphe G_2 est la restriction de la fonction de circulation admissible F dans le graphe G_2 .

R e m a r q u e 1. Réciproquement, deux circulations standards admissibles $(\bar{c}/\bar{b}', F')$ dans le graphe G_1 et $(\bar{b}'/\bar{c}, F'')$ dans le graphe G_2 (Fig. 2) peuvent être étendus à une circulation admissible $(\bar{c}\bar{b}'/\bar{b}'\bar{c}, F)$ d'un graphe escalier, si $F'_{10} = F''_{20}$.

Fig. 3 — G'^f avec $f = \bar{c}\bar{b}'$.

D é m o n s t r a t i o n. Nous transformons le graphe G' (Fig. 2) en le graphe G (Fig. 1) comme suit: nous détachons l'arrête P_{10} (P_{20}) dans le sommet x_2 (x_1) et nous le collons dans le sommet x_1 (x_2). On obtient le graphe G (Fig. 1) avec la circulation admissible: $F_{10} = F'_{10} = F''_{20} = F_{20}$; $F = F'$ par les arrêtes de G_1 et $F = F''$ par les arrêtes de G_2 .

T h é o r è m e 2. Pour toute circulation admissible $(\bar{c}\bar{b}'/\bar{b}'\bar{c}, F)$ dans un graphe escalier (Fig. 1) il existe une circulation admissible $(\bar{c}\bar{b}', F')$ dans un graphe éventail (Fig. 3) [2].

D é m o n s t r a t i o n. Le graphe donné est transformée dans un graphe éventail en condensant les arrêtes P_{2r}^h avec $r = \begin{cases} i = \overline{1, t}, h = 1, \\ j = \overline{1, n}, h = 2, \end{cases}$ et P_{20} dans le sommet O_2 , en résultant le graphe éventail G'^f avec $f = \bar{c}\bar{b}'$ (Fig. 3).

Les fonctions de capacité c' et C' pour le graphe éventail sont définies ainsi:

$$\left. \begin{aligned} c'_{1r} &= \max_k c_{kr}^h \\ C'_{1r} &= \min_k C_{kr}^h \end{aligned} \right\} \text{ pour } k = 1, 2 \text{ et } r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2; \end{cases}$$

$$c'_{10} = \max(c_{10}, c_{20}), \quad C'_{10} = \min(C_{10}, C_{20}),$$

$$c'_s = \bar{c}_s, \quad C'_s = \bar{C}_s, \quad \text{pour } s = t, n;$$

$$c_r^h = c_r^h \text{ et } C_r^h = C_r^h, \quad \text{pour } r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2. \end{cases}$$

On définit une circulation admissible $(\bar{c}\bar{b}', F)$ dans le graphe $G^{\bar{c}, \bar{b}'}$ comme la restriction de la fonction de circulation admissible F dans le graphe donné. Pour toute arête du graphe éventail $F' = F$; donc les équations de conservation sont satisfaites.

Dans le sommet O_2 nous avons [5] $\bar{F}_t + \bar{F}_n = \sum (F_j^2; 1 \leq j \leq n) + \sum (F_i^1; 1 \leq i \leq t)$.

Remarque 2. Réciproquement, une circulation admissible $(\bar{c}\bar{b}, F)$ dans un graphe éventail G' (Fig. 3) peut être étendu à une circulation admissible $(\bar{c}\bar{b}'/\bar{b}'\bar{c}, F)$ d'un graphe escalier.

Démonstration. Nous scindons le sommet O_2 dans les arêtes

$$P_{2r}^h, \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2 \end{cases} \quad \text{et} \quad P_{20}.$$

Nous définissons les fonctions de capacité sur les nouvelles arêtes:

$$\begin{aligned} c_{20} &= 0, \quad C_{20} = \infty, \\ c_{2r}^h &= 0, \quad C_{2r}^h = \infty, \end{aligned} \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2. \end{cases}$$

On peut facilement vérifier que pour une circulation admissible $(\bar{c}\bar{b}, F')$ dans un graphe éventail G' (Fig. 3) peut être étendu à une circulation admissible $(\bar{c}\bar{b}'/\bar{b}'\bar{c}, F)$ d'un graphe escalier G ainsi:

$$\begin{aligned} F_{20} &= F'_{10}, \\ F_{2r}^h &= F'_{1r}^h = F_{1r}^h, \end{aligned} \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2. \end{cases}$$

Théorème 3. Pour toute circulation admissible $(\bar{c}\bar{b}'/\bar{b}'\bar{c}, F')$ dans un graphe escalier (Fig. 1) il existe une circulation admissible $(\bar{b}'\bar{c}, F'')$ dans un graphe éventail (Fig. 4) [2].

Le théorème on peut démontrer en condensant les arrêtes

$$P_{1r}^h, \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2 \end{cases} \quad \text{et} \quad P_{10} \quad \text{dans le sommet } O_1.$$

Remarque 3. Réciproquement, une circulation admissible $(\bar{b}'\bar{c}, F'')$ dans le graphe éventail G' (Fig. 4) peut être étendu à une circulation admissible $(\bar{c}\bar{b}'/\bar{b}'\bar{c}, F)$ d'un graphe escalier.

Démonstration. Nous scindons le sommet O_1 dans les arrêtes

$$P_{1r}^h, \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2 \end{cases} \quad \text{et} \quad P_{10}.$$

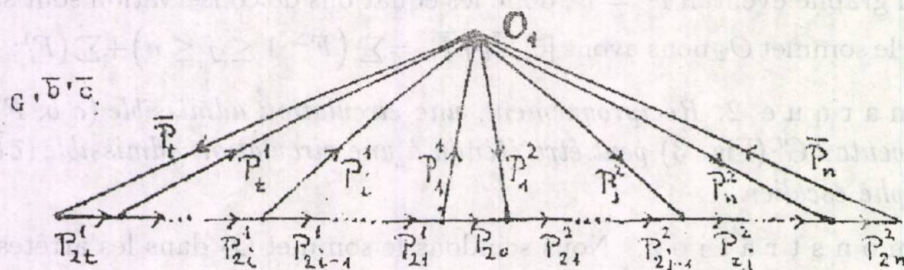


Fig. 4

Nous définissons les fonctions de capacité et de circulation tout comme dans la Remarque 2.

Théorème 4. Pour toute circulation admissible de type $(\bar{c}\bar{b}'/\bar{b}'\bar{c})$ dans un graphe escalier (Fig. 1) il existe une circulation admissible dans les graphes circuits.

Démonstration. Le graphe donné peut être écrit comme somme directe de deux graphes escalier disjointes- sommets (Théorème 1). En vertu du Théorème 2 [4] il résulte le Théorème 4.

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BIBLIOGRAPHIE

1. Ford L. R., Fulkerson D. R., *Flots dans les graphes*. Gauthier-Villars, 1967.
2. Ghiuș V., *Les types de circulation dans les graphes éventails*. Bul. Inst. Polit. Iași, XXXI (XXXV), 1-2, s. I (1985).

3. G h i u ș V., *La circulation standard admissible dans un graphe latticiel avec une dimension* (en cours d'apparition).
4. G h i u ș V., *Transformation de la circulation standard admissible dans un graphe latticiel avec une dimension non orienté*. Bul. Inst. Polit. Iași, **XXXVIII (XLII)**, 1 - 4, s. I, 60-63 (1992).
5. G h i u ș V., *Les types de circulations admissibles dans un graphe escalier non orienté*. Bul. Inst. Polit. Iași, **XLIII (XLVII)**, 1 - 2, s. I (1997).

TIPURI DE CIRCULAȚII ADMISIBILE ÎNTR-UN GRAF SCARĂ NEORIENTAT (VI)

(Rezumat)

Se studiază descompunerea unei circulații admisibile de tip $(\bar{c}\bar{b}'/\bar{b}'\bar{c})$ dintr-un graf scară în circulații standard admisibile din grafuri de aceeași formă și reciproc. De asemenea sus amintita circulație admisibilă este transformată în circulații admisibile de tip $\bar{b}'\bar{c}(\bar{c}\bar{b}')$ din grafuri evantai și în circulații admisibile de tip circuit.

**A SYSTEM OF HIDDEN MARKOV MODELS
FOR AUTOMATIC SPEECH RECOGNITION
A SPEAKER-INDEPENDENT APPLICATION**

BY

GUSTAVO SANTOS-GARCÍA and C. CĂLIN

Abstract. A hybrid system of automatic recognition for isolated Spanish words, independent of the speaker, is developed using Hidden Markov Models (HMM) and Vector Quantization (VQ). An exhaustive review of the working and operability of the HMM is carried out. We have used the mathematical environment MatLab for the computer development. In the system proposed we performed a training stage for the models with sound samples coming from several speakers (half of which were men and the other half women). For testing the models we used another group of speakers different from those of the training stage. Finally, the results obtained with this recognition method are given.

Key words: automatic speech recognition, hidden Markov models, vector quantization.

1. Introduction

Speech recognition is a very complex problem. The complex and variable way in which language messages are codified makes it very difficult. Many government-funded speech research projects for the military sector have bolstered the growth of the speech industry as a whole. Technical improvements made under military auspices are being applied elsewhere in a host of commercial applications.

In the field of speech a lot of studies have been carried out by many researchers for many years now. These researchers have been encouraged by the huge development of computers in speed, storage, interfaces, etc. In this area we can find several applications that are very different in foundation and performance: storage and digital transmission of the speech signal, synthesis of speech, speaker identification and verification speech recognition, handicappers aid, improvement of signal quality, etc.

Looking at the shape of the vocal tract, which basically and simply is a cylindrical pipe closed on one side by the vocal cords and opened on the other side by the lips, allows us to set up a model of speech production based on principles of digital signal processing. A simplified picture of the human vocal tract is represented in Fig. 1.

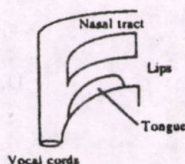


Fig. 1 — Scheme of human vocal tract.

In speech recognition there are two great fields [6]: *isolated speech recognition* and *continuous speech recognition*. This study belongs to the first one and is made with independence of the speaker. A discrete set of words or phonemes forms the vocabulary of the recognizer. The recognizing system is shown in Fig. 2. Finally a sound is given to the recognizer and it will assign an element of the vocabulary to this sound.

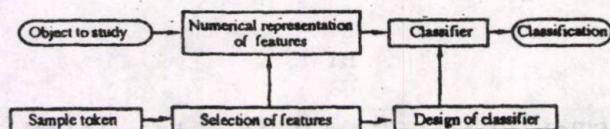


Fig. 2 — Scheme of a recognizing system.

In Continuous Speech Recognizing (CSR), a sentence or text is pronounced. Therefore, there is the added problem of finding where the word starts and finishes. It will be necessary to determine the silence and also the problem of recognizing every word as before.

There are many techniques for speech recognition. Some of them are: Hidden Markov Models [4], [5], [10], Pattern Recognition, Statistics Recognizers [12], Neural Networks [7], [13], etc. These general purpose recognizers take speech data through the speech signal. The sources of speech yield and the shape of the vocal tract are relatively independent [9]. Most hybrid models for continuous speech recognition or isolated word recognition have fixed architecture during training, but it is possible to implement a self-structuring hidden control neural model for isolated word recognition [14]. A powerful system is the SRI-DECIPHER system which is a phone-based, speaker-independent, continuous-speech recognition system, based on semicontinuous HMMs [2] developed at SRI-International.

From these data it is possible to obtain, through digital signal processing techniques, either a special codification or a parameter series. These can be divided in a temporal and spectral method depending on the methods used to obtain them (fundamental frequency, formant frequencies, energy per band, etc.).

2. Hidden Markov Models

We will use Hidden Markov Models as classifiers. A HMM needs a primary training stage to work properly. In this stage the initial parameters of the model are

fitted to the real data. Later on the model can be use directly and system efficiency is then evaluated [11]. This process is shown in Figs. 3 and 4.

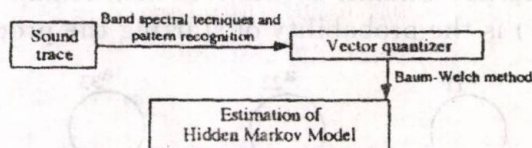


Fig. 3 — Scheme of Hidden Markov Model training.

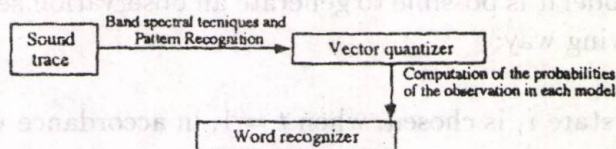


Fig. 4 — Scheme of Hidden Markov Model test.

The number of states of the model will be denoted by N . The definition of state depends on the specific problem to study, but every definition has one thing in common, which is that the signal in a state has something measurable, i.e., certain distinguishable properties. In speech recognition the states correspond to a temporal sequence of situations, which we can associate experimentally with temporal parts of the sound.

At each time t the process will come into a state, which is based on a distribution of transition probabilities depending on previous state (property of markovian processes). It is interesting to remark that transition may be such that the process remains at the previous state.

When transition has been performed, observation of the symbol shall be carried out according to a probability distribution depending only on the actual state. This probability distribution will be fixed for every state independently of the moment and the way in which it has arrived at that state. Therefore we will have N observation probability distributions representing random variables or stochastic processes. T will denote the observation sequence length, that is, the number of observations in every trace analyzed. M will be the possible symbols that can be observed. Therefore, we have a set of states that will be denoted by $Q = \{q_1, q_2, \dots, q_N\}$. The discrete set of possible symbols will be $V = \{v_1, v_2, \dots, v_M\}$. The $N \times N$ transition matrix between states will be $A = \{a_{ij}\}$, where $a_{ij} = P(q_j \text{ at time } t+1 | q_i \text{ at time } t)$ is the probability of reaching state q_j from state q_i , that is, not depending on t since it is a markovian process. In Fig. 5 we have a scheme of a very simple case.

The probability distribution of observable symbols at state j will be carried out by the M dimensional vector $B_j = \{b_j(k)\}$, where $b_j(k) = P(v_k \text{ at } t | q_j \text{ at } t)$ is the

probability of choosing the symbol v_k being at state q_j , that is not depending on time. With these vectors we form the $N \times M$ matrix B . In order to complete the model we must give an initial distribution of the states, that will be denoted, as in the general theory of Markov Chains, by the N dimensional vector $\pi = \{\pi_i\}$, where $\pi_i = P(q_i \text{ at time } t = 1)$ is the probability of starting the process at state q_i .

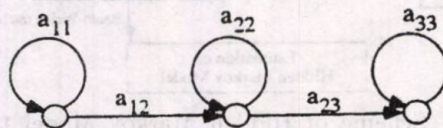


Fig. 5 — Acoustic model topology.

Using the model it is possible to generate an observation sequence $O = O_1, O_2, \dots, O_T$ in the following way:

1. An initial state i_1 is chosen, when $t = 1$, in accordance with initial distribution π ;
2. An observation O_t is chosen in accordance with the probability distribution of the symbols at state i_t , $b_{i_t}(k)$, for $k = 1, \dots, M$;
3. A new state i_{t+1} is chosen from state i_t in accordance with the probability distribution of transition $\{a_{i_t i_{t+1}}\}$, where $i_{t+1} = 1, \dots, N$;
4. Making $t = t + 1$ the process turns to 2 while $t < T$; otherwise the process is concluded.

From now on we will use notation $\lambda = (A, B, \pi)$ for a Hidden Markov Model (HMM). For the specification of a HMM it is necessary to choose the number N of states, the set of symbols V (in this work a discrete set of symbols with M elements will be used, although the non-discrete case could be studied also in a more general theory) and the three probability densities A , B and π . For most of the applications the π vector, which represents the initial conditions of the model, is the one of least importance. The most relevant distribution is, in general, matrix B since it is directly referred to the observed symbols. On the other hand we can say that matrix A is important in certain problems, but not very important in others, such as the case considered in this paper – recognition of isolated words–.

3. Computation of the Probability of an Observation for a Hidden Markov Model

The first problem is to obtain the probability of a certain observation sequence $O = O_1, O_2, \dots, O_T$ for a given model $\lambda = (A, B, \pi)$. The problem can be seen as the evaluation model for a specific observation. This point of view is very useful because we want to choose a model among several possible models in order to fit it better to a given observation. We will now compute this probability.

The shortest way to make this computation is to consider every possible sequence of states with length T . For each of these sequences $I = i_1 i_2 \dots i_T$, the probability of the observation sequence $O = O_1, O_2, \dots, O_T$ is the following:

$$P(O|I, \lambda) = b_{i_1}(O_1) \dots b_{i_T}(O_T).$$

On the other hand, the probability of each sequence of states is the following:

$$P(I|\lambda) = \pi_{i_1} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{T-1} i_T},$$

and using the total probability rule,

$$\begin{aligned} P(I|\lambda) &= \sum_I P(O|I, \lambda) P(I|\lambda) = \\ &= \sum_{i_1, i_2, \dots, i_T} \pi_{i_1} b_{i_1}(O_1) a_{i_1 i_2} b_{i_2}(O_2) a_{i_2 i_3} \dots a_{i_{T-1} i_T} b_{i_T}(O_T). \end{aligned}$$

The physical interpretation of the last formula is the following: fixing a sequence of states I , initially at time $t = 1$, the process is at state i_1 with probability π_{i_1} , and the symbol O_1 is generated with probability $b_{i_1}(O_1)$. Then, a transition to state i_2 shall be made with probability $a_{i_1 i_2}$, and the symbol O_2 shall be chosen with probability $b_{i_2}(O_2)$. The process continues in this way until the last transition from state i_{T-1} to state i_T with probability $a_{i_{T-1} i_T}$, where the symbol O_T is chosen with probability $b_{i_T}(O_T)$. The total probability is the result of summing these terms for every possible sequence of states.

This is very easy to verify, the computation of this probability needs $(2T - 1)N^T$ multiplications, because every term of the sum has $2T - 1$ factors and the total number of terms of the sum are the possible variations with repetition of T states among N . Moreover it is necessary to make $N^T - 1$ sums. This computation could be an overflow, even for low values of N and T . For example, if $N = 5$ and $T = 100$ about $2 \cdot 100 \cdot 5^{100} \approx 10^{72}$ operations will be required. Therefore it is necessary to look for a more efficient process for the computation of this probability. For this purpose the following *forward-backward* process will be used.

Let us consider the *forward-backward* variable,

$$\alpha_t(i) = P(O_1, O_2, \dots, O_t, i_t = q_i | \lambda),$$

that is, the probability of a partial observation sequence until time t and the state q_i at time t , for a given model λ . We can compute these probabilities recurrently as follows:

1. $\alpha_1(i) = \pi_i b_i(O_1)$ for $1 \leq i \leq N$;
2. For each $t = 1, 2, \dots, T - 1$ and $1 \leq j \leq N$ the recurrency law, $\alpha_{t+1}(j) =$

$$= \left[\sum_{i=1}^N \alpha_t(i) a_{ij} \right] b_j(O_{t+1})$$
 is true;
3. Then $P(O|\lambda) = \sum_{i=1}^N \alpha_T(i)$.

In the first step the probability of state q_i appears and the initial observation O_1 , which has been calculated using the conditional probability definition. In the second step the probability is computed according to the form of passing to state q_j at time $t + 1$ from the possible states q_i , for $i = 1, 2, \dots, N$ at time t . On the one hand $\alpha_t(i)$ is the probability of $O_1, O_2, \dots, O_t$ and state q_i at which the process remains at time t . Hence the product $\alpha_t(i)a_{ij}$ will be the probability of $O_1, O_2, \dots, O_t$ and state q_j at which the process remains at time $t + 1$ assuming that the process was at state q_i at time t . Summing this product over the N possible states q_i , for $i = 1, 2, \dots, N$, at time t , the probability of q_j at time $t + 1$ is obtained with all previous observations. Thus, the probability $\alpha_{t+1}(j)$ will be the product of the last probability multiplied by $b_j(O_{t+1})$. Finally, in the third step the probability $P(O|\lambda)$ is obtained as the sum of the probabilities $\alpha_T(i)$, since, $\alpha_T(i) = P(O_1, O_2, \dots, O_T, i_T = q_i|\lambda)$.

The computation of each $\alpha_t(j)$, $1 \leq t \leq T$, $1 \leq j \leq N$, demands a number of operations of the order of N^2T , whereas the number of operations in the direct way were of the order of $2TN^T$. Precisely $N(N+1)(T-1) + N$ multiplications and $N(N-1)(T-1)$ sums. In the case of $N = 5$ and $T = 100$ only about 3000 operations will be necessary whereas 10^{72} operations necessary in the direct way. This means a saving of 10^{69} operations with this method.

The key of this process is that there are only N states. Thus every sequence of states links these N states, the sequence length not being relevant. We have implemented this process with MatLab,

```
% Computation of alphas
% alpha t(i), 1 <= i <= N, 1 <= t <= T
alpha=zeros (T, N); % initialization of alpha
alpha= (1, :) = pi' * B(O(1), :);
for t = 1 : T - 1,
    for j = 1 : N,
        alpha (t + 1, j) = (alpha (t, :) * A(:, j)) * B(O(t + 1), j);
    end,
end,
```

First the alpha matrix is initialized and then the correspondent values for each element of this matrix are given in a recurrent way. Thus, the variables for $t = 1$ are defined in the first step. Afterwards values for the remaining t are given through a loop. $O(t)$ represents the observation sequence taken of the observations performed for several speakers. This sequence is introduced in each step of the process and enables the model to be improved step by step. The matrix $A(i, j)$ is the transition matrix and $B(i, j)$ the probability of each symbol at each state. These matrices are introduced initially and at each step of the process they are modified. The possibility of using the matrix product facilitates the assignation of these variables. The probability of each observation sequence is,

```
% Computation of P(O|lambda)
PO=sum(alpha(T,:));
```

In a similar way we can now consider the following *backward* variable, $\beta_t(i) = P(O_{t+1}, O_{t+2}, \dots, O_T | i_t = q_i, \lambda)$, that is, the probability of a partial sequence of

observations from time $t + 1$ to the end, given a state q_i at time t and in the model λ . As before, we can compute the probabilities in a recurrent way:

1. $\beta_T(i) = 1$, $1 \leq i \leq N$;
2. For $t = T - 1, T - 2, \dots, 1$, and $1 \leq i \leq N$ we have

$$\beta_t(i) = \sum_{j=1}^N a_{ij} b_j(O_{t+1}) \beta_{t+1}(j).$$

The computation of $\beta_t(i)$ for $1 \leq t \leq T$, $1 \leq i \leq N$, needs – as in $\alpha_t(i)$ – a number of operations of the order of N^2T .

The MatLab program we have developed for computing these variables is the following:

```
% Computation of betas
% beta t(i), 1 <= i <= N, 1 <= t <= T
beta=zeros (T,N); % initialization of beta
beta(T,:)=ones(1,N);
for t = T - 1 : -1 : 1,
    beta (t,:) = (B(O(t+1),:).* beta (t+1,:)) * A';
end,
```

where the initial values of the vector are introduced and then, with a loop, the elements of the variable beta are introduced. The matrices A and B are involved in this definition as in the case of alpha.

4. Fitting the Model with the Maximum Probability

Finally, we try to look for the parameters of the model $\lambda = (A, B, \pi)$ making the probability $P(O|\lambda)$ the maximum, i.e., search will be made for the model best fitted to the observed sequence. For this reason we will say that it is a *training* sequence, because it is used for training the model. The training problem is very important for most of the applications of HMM, because it makes it possible to optimize the parameters of the model for the observed data, i.e., to create the best models for real phenomena. Nevertheless this problem has no easy solution. In fact, no analytic process for finding the model of maximum probability is known. For this reason it will be necessary to use algorithms like the *B a u m-W e l c h* algorithm or gradient techniques. We will now discuss the iterative process, with which the physical meaning of estimation of the parameters can be seen.

We shall start with the following definition, $\xi_t(i, j) = P(i_t = q_i, i_{t+1} = q_j | O, \lambda)$, which is the probability of a path passing through state q_i at time t , and a transition of this state to state q_j occurs at time $t + 1$, given an observation sequence for a specific model. Therefore, this probability can be written as follows,

$$\xi_t(i, j) = \frac{\alpha_t(i) a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)}{P(O|\lambda)},$$

where $\alpha_t(i)$ takes into account the first t observations, the term $a_{ij}b_j(O_{t+1})$ refers to transition to state q_j at time $t+1$ and the observation of the symbol O_{t+1} , while the term $\beta_{t+1}(j)$ makes reference to the rest of the observation sequence. $P(O|\lambda)$ is the normalization factor that makes the product exactly the probability $\xi_t(i, j)$.

We define $\gamma_t(i) = P(i_t = q_i | O, \lambda)$, which is the probability of the process being at state q_i at time t , given the observation sequence O for a specific model. Thus it is possible to relate $\gamma_t(i)$ and $\xi_t(i, j)$ using the following sum of these probabilities

$$\text{over } j, \gamma_t(i) = \sum_{j=1}^N \xi_t(i, j).$$

The sum of $\gamma_t(i)$ over t can be interpreted as the expected value for the number of times the process passes through state q_i , or what is the same, the expected number of transitions made from state q_i , excluding the last time T , because from the last state there is no possible transition. Analogously, the sum of $\xi_t(i, j)$ from $t = 1$ to $t = T - 1$ can be understood as the expected number of transitions from state q_i to state q_j , i.e.,

$$\sum_{t=1}^{T-1} \gamma_t(i) = \text{Expected number of transitions from } q_i ,$$

$$\sum_{t=1}^{T-1} \xi_t(i, j) = \text{Expected number of transitions from } q_i \text{ to } q_j .$$

With these formulae we can use the Baum-Welch method for the re-estimation of the parameters of HMM. The re-estimation formulae for π , a and B are the following:

$$1. \bar{\pi}_i = \gamma_1(i), \quad 1 \leq i \leq N;$$

$$2. \bar{a}_{ij} = \frac{\sum_{t=1}^{T-1} \xi_t(i, j)}{\sum_{t=1}^{T-1} \gamma_t(i)}, \quad 1 \leq i, j \leq N;$$

$$3. \bar{b}_j(k) = \frac{\sum_{t=1; O_t=k}^{T-1} \gamma_t(j)}{\sum_{t=1}^{T-1} \gamma_t(j)}, \quad 1 \leq j \leq N, \quad 1 \leq k \leq M.$$

The estimation formula for π_i , is easily the probability of being at state q_i at time $t = 1$. The re-estimation formula for a_{ij} , is the ratio between the expected number of transitions from state q_i to state q_j and the expected number of transitions from state q_i (for this reason the sum is made only until $T - 1$). In the same way, the re-estimation of $b_j(k)$ is the ratio between the expected number of times the process stays at state j and the symbol k is observed, and the expected number of times the process stays at state j . We have used, to some extent, the frequency concept of probability as a ratio between absolute frequency and total frequency.

If we take an initial model λ and the re-estimation $\bar{\lambda}$ of that model, then it is possible to say that, either the initial model λ defines a critical point of likelihood function, that is $\bar{\lambda} = \lambda$, or, the model $\bar{\lambda}$ is more likely in the sense that $P(O|\bar{\lambda}) > P(O|\lambda)$, i.e., we have found a model $\bar{\lambda}$ verifying that the observation sequence is yielded with greater probability.

Therefore, using $\bar{\lambda}$ instead of λ in an iterative way and repeating the last re-estimations we can improve the probability of O in the obtained model step by step until the process achieves a determined limit point. The final result is the estimated model we wish to find.

5. Conclusions

1. Summarizing the work we have considered the following speech recognition scheme: we design a HMM with N states for every word of a given vocabulary V . Using techniques of Vectorial Quantization (VQ), we can show the speech signal as a sequence of symbols of a code VQ derived from a set of codes [10], [12]. In this way we start with a training sequence for each word of the vocabulary, that is a series of repetitions of the spoken word (by one or more speakers). We use the solutions calculated before for optimizing the parameters of the model for every word. For understanding the physical sense of states of the model we will use the solution given by the well-known Viterbi algorithm for segmenting the training sequence of each word into states, and will thus be able to study the observations in each state. The result of this study can be the improvement of the model. Finally, for recognizing a word, the solution provided by the computation of the probability of an observation for a given model will be used in order to *score* the model of every word from the observation sequence performed and the word with the greatest score will be chosen.

2. We have considered a vocabulary V of eight Spanish words. Specifically: $V = \{\text{equipo, copa, petaca, tabaco, bigote, duda, bate, vida}\}$. In this set several Spanish words shared occlusive consonants and the Spanish vowels. In every performance we have taken the set of observations from a total set of 11 symbols according to spectral bands techniques. The set of symbols is: $\{p, t, k, b, d, g, a, e, i, o, u\}$.

For the training phase ten different speakers were taken to pronounce each word of the vocabulary three times. This provided a total of $10 \times 8 \times 3$ sound traces. When the system had been trained it was time for evaluation. For that we took samples of another nine different speakers – they were not the same speakers as those of the training phase –. They pronounced each word of the vocabulary three times obtaining a total of $9 \times 8 \times 3$ sound traces.

3. Analysis of the results is shown in Fig. 6. In this figure we show the percentages of prediction for each of the eight words. Given a word, for instance *equipo*, our recognizing system has guessed the right word in 74.0741 % of the cases, while in 0 % it has given the word *copa*, 3.7037 % the word *petaca*, 0 % the word *tabaco*, 22.2222 %

the word *bigote*, 0 % the word *duda*, 0 % the word *bate* and, finally, 0 % the word *vida*.

The results of Hidden Markov Models we have developed are similar, in the sense of scoring, to those obtained in the different neuronal nets models, except with the multilayer perceptron nets model with which the results are rather better [1], [13].

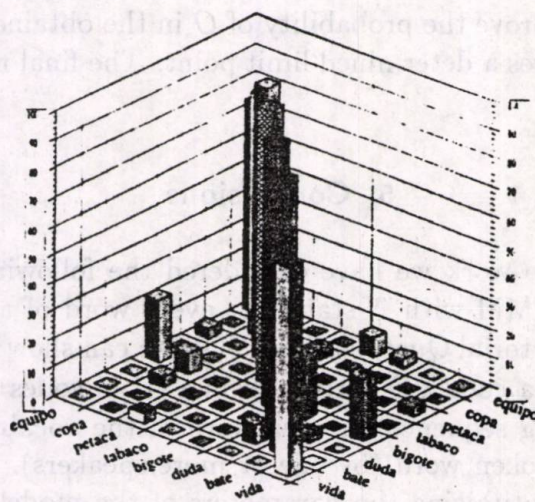


Fig. 6 — Results for recognition of vocabulary formed by 8 Spanish words, through Hidden Markov Models.

4. Some basic speech signal processing problems still remain. Sensitivity to background acoustic noise, changes in the channel characteristics, and misrecognitions of out-of-vocabulary responses all remain difficult problems in speech recognition applications. As our ability to represent a real acoustic signal increases, the need to reject spurious input becomes more acute. While the techniques work well in noise-free environments, demonstration of high performance on even the simplest tasks in operational environments remains a challenge.

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REFERENCES

1. Boulard H., Wellekens Ch. J., *Links Between Markov Models and Multilayer Perceptrons*. IEEE Trans. Pattern Analysis and Machine Intelligence, **12**, 1167-1178 (1990).
2. Cohen M., Murveit H., Bernstein J., Price P., Weintraub M., *The DECIPHER Speech Recognition System*. ICASSP, Albuquerque, New Mexico, 1990, pp. 77-80.

3. Cohen M., Franco H., Morgan N., Rumelhart D., Abrash V., *Context-Dependent Multiple Distribution Phonetic Modelling with MLPs*. NIPS-5, 1992.
4. Huang X. D., *Phoneme Classification Using Semicontinuous Hidden Models*. IEEE Trans. Signal Processing, **40**, 5, 1062-1067 (1992).
5. Kenny P., Lennig M., Mermelstein P., *Speaker Adaptation in a Large-Vocabulary Gaussian HMM Recognizer*. IEEE Trans. Pattern Analysis and Machine Intelligence, **12**, 917-920, 1167-1178 (1990).
6. Levinson S. E., *Structural Methods in Automatic Speech Recognition*. Proceedings of the IEEE, **73**, 11, 1625-1650 (1985).
7. Lippman R. P., *An Introduction to Computing with Neural Nets*. IEEE ASSP Magazin, 4-22 (April 1987).
8. Makhoul J., Roucos S., Gish H., *Vector Quantization in Speech Coding*. Proceedings of the IEEE, **73**, 11, 1551-1588 (1985).
9. Rabiner L. R., Shafer R., *Digital Processing of Speech Signals*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1978.
10. Rabiner L. R., Levinson S. E., Sondhi M. M., *On the Application of Vector Quantization and Hidden Markov Models to Speaker-Independent, Isolated Word Recognition*. The Bell System Technical Journal, **62**, 4, 1075-1105 (1983).
11. Rabiner L. R., *Mathematical Foundations of Hidden Markov Models*. AT&T Bell Laboratories, NATO ASI Series, **V-46**, 186-205 (1988).
12. Santos-García G. et al., *Three Pattern Recognition Models for Spanish Vowels in a Multispeaker Environment*. Proceedings of the Ninth IASTED International Symposium, Innsbruck, 1990, pp. 379-382.
13. Santos-García G., *The Hopfield and Hamming Networks Applied to the Automatic Speech Recognition of the Five Spanish Vowels*. Proceedings of Intl. Conf. on Neural Nets and Genetic Algorithms, Innsbruck, 1993.
14. Sorensen H. B. D., *Triphone- and Word-based Self- Structuring Hidden Control Models*. IEEE Intl. Conf. on Neural Networks, San Francisco, 1993.

UN SISTEM DE MODELE MARKOV ASCUNSE PENTRU RECUNOAȘTEREA AUTOMATĂ A VORBIRII

O aplicație independentă de vorbitor

(Rezumat)

Se studiază recunoașterea unor cuvinte speciale spaniole cu ajutorul modelului Markov ascuns folosind programul MatLab pentru calculele matematice. Pentru testarea modelului s-a recurs la diferite grupuri de cunoscători ai limbii spaniole din diverse regiuni.

AN EXACT SOLUTION TO FOUCAULT'S PENDULUM PROBLEM

BY

DANIEL CONDURACHE and
MIHAELA-HANAKO MATCOVSCHI

Abstract. This paper presents an exact solution to the Foucault's pendulum problem. The solution is obtained by using the symbolic representation of vectorial functions through matrices. The main result of this paper is contained into Theorem 1 and allows to imagine the particle trajectory.

Key words: Foucault's pendulum, symbolic representation, turn-tensor.

1. Introduction

The aim of this paper is to introduce an exact solution to the Foucault's pendulum problem not yet signalled. The study of the Cauchy problem describing the particle motion uses the symbolic representation of vectorial functions through matrices presented in [4]. A brief presentation of the main results involved in this paper follows.

Let $\mathcal{V}_3$ be the linear space of free vectors, $\mathbf{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ an orthonormal right oriented basis in this space, and $\mathbf{V}_3$ the linear space of column matrices with 3 real numbers elements.

The application defined by

$$(1) \quad \Lambda: \mathcal{V}_3 \rightarrow \mathbf{V}_3, \quad \Lambda(\mathbf{v}) = \mathbf{v},$$

where $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, $v_k \in \mathbb{R}$, $k = \overline{1, 3}$ is an arbitrary element of $\mathcal{V}_3$ and $\mathbf{v} = [v_1 \ v_2 \ v_3]^T$ the corresponding element of $\mathbf{V}_3$, is an isomorphism of linear spaces.

To each free vector $\boldsymbol{\omega} \in \mathcal{V}_3$ defined by $\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3$ the antisymmetric matrix

$$(2) \quad \tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

can be associated. Denote by $\omega = \text{vect}(\tilde{\omega})$.

For a given free vector $\omega \in \mathcal{V}_3$ the isomorphism introduced by (1) satisfies the condition

$$(3) \quad \Lambda(\omega \times v) = \tilde{\omega} \cdot v$$

for every $v \in \mathcal{V}_3$. Since application Λ is bijective, from relation (3) it follows that

$$(4) \quad \begin{aligned} \Lambda^{-1}(\tilde{\omega} \cdot v) &= \omega \times v, \\ \Lambda^{-1}(\tilde{\omega}^2 \cdot v) &= \omega \times (\omega \times v). \end{aligned}$$

Denote by $\mathcal{V}_3^* = \{f : I \subseteq \mathbb{R} \rightarrow \mathcal{V}_3\}$ the linear space of vectorial functions of real variable and by $\mathbf{V}_3^* = \{f : I \subseteq \mathbb{R} \rightarrow \mathbf{V}_3\}$ the linear space of column matrix functions of real variable. The symbolic representation defined by (1) induces the representation

$$(5) \quad \Lambda^* : \mathcal{V}_3^* \rightarrow \mathbf{V}_3^*, \quad \Lambda^*(v(t)) = v(t),$$

where $v(t) = v_1(t)e_1 + v_2(t)e_2 + v_3(t)e_3$ is an arbitrary element of $\mathcal{V}_3^*$ and $v(t) = [v_1(t) \ v_2(t) \ v_3(t)]^T$ the corresponding element of $\mathbf{V}_3^*$. Application (5) is an isomorphism of linear spaces.

The same way we have proceeded for free vectors, to each vectorial function $\omega \in \mathcal{V}_3^*$ defined by $\omega(t) = \omega_1(t)e_1 + \omega_2(t)e_2 + \omega_3(t)e_3$ the antisymmetric matrix function

$$(6) \quad \tilde{\omega}(t) = \begin{bmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{bmatrix}, \quad t \in I,$$

can be associated. Denote by $\omega = \text{vect}(\tilde{\omega})$.

For a given vectorial function $\omega \in \mathcal{V}_3^*$, the isomorphism introduced by (5) satisfies the condition

$$(7) \quad \Lambda^*(\omega \times v) = \tilde{\omega} \cdot v$$

for every vectorial function $v \in \mathcal{V}_3^*$. Since application Λ^* is bijective, relation (7) implies that

$$(8) \quad \begin{aligned} \Lambda^{*-1}(\tilde{\omega} \cdot v) &= \omega \times v, \\ \Lambda^{*-1}(\tilde{\omega}^2 \cdot v) &= \omega \times (\omega \times v). \end{aligned}$$

Let $\omega \in \mathcal{V}_3^*$ be a vectorial function of fixed direction

$$(9) \quad \omega(t) = \omega(t)u,$$

where u is a constant vector, $|u| = 1$, and $\omega : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ a derivable real function of real variable. The corresponding antisymmetric matrix-function $\tilde{\omega}(t)$ has the form

$$(10) \quad \tilde{\omega}(t) = \omega(t)\tilde{u}, \quad t \in I,$$

where $\tilde{u}$ is the antisymmetric matrix corresponding to vector u . Matrix $\tilde{\omega}(t)$ satisfies the condition

$$(11) \quad \tilde{\omega}(t_1) \cdot \tilde{\omega}(t_2) = \tilde{\omega}(t_2) \cdot \tilde{\omega}(t_1), \quad \forall t_1, t_2 \in I.$$

2. Foucault's Pendulum Problem

The famous experience performed by Léon Foucault at the Panthéon in Paris in 1852 revealed Earth's rotation around the pole-axis. The theoretical importance of this experiment is obvious. The experiment modelling leads to the study of an harmonic oscillator with respect to a rotating frame around a fixed direction.

The mathematical model of this experiment [3] is represented by the Cauchy problem:

$$(12) \quad \begin{cases} \ddot{\mathbf{r}} + 2\dot{\boldsymbol{\omega}} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \omega_*^2 \mathbf{r} = \mathbf{0}, \\ \mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(0) = \mathbf{v}_0, \end{cases}$$

where $\mathbf{r}$ is the position vector with respect to a fixed frame of reference with origin in the relative equilibrium position $\mathbf{r} = \mathbf{0}$; $\boldsymbol{\omega}(t)$ is the earth angular velocity, considered to be a vectorial function of fixed direction ($\boldsymbol{\omega}(t) = \omega(t)\mathbf{u}$, with $\mathbf{u}$ a constant vector satisfying $|\mathbf{u}| = 1$ and $\omega : [0, +\infty) \rightarrow \mathbb{R}$ a derivable real function); $\omega_* = g/l$, with g – gravitational acceleration, l – length of pendulum.

In order to solve problem (12), it is convenient to apply the symbolic representation introduced by (5), leading to the equivalent matrix Cauchy problem:

$$(13) \quad \begin{cases} \ddot{\mathbf{r}} + 2\tilde{\boldsymbol{\omega}} \cdot \dot{\mathbf{r}} + \tilde{\boldsymbol{\omega}}^2 \cdot \mathbf{r} + \dot{\tilde{\boldsymbol{\omega}}} \cdot \mathbf{r} + \omega_*^2 \mathbf{r} = \mathbf{0}, \\ \mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(0) = \mathbf{v}_0. \end{cases}$$

Consider $\boldsymbol{\xi}$ the column matrix with complex functions elements defined by

$$(14) \quad \boldsymbol{\xi} = \mathbf{r} - \mathbf{j} \frac{\dot{\mathbf{r}} + \tilde{\boldsymbol{\omega}} \cdot \mathbf{r}}{\omega_*}, \quad \mathbf{j}^2 = -1.$$

Taking into account (13), we obtain that $\boldsymbol{\xi}$ is the solution to the Cauchy problem:

$$(15) \quad \begin{cases} \dot{\boldsymbol{\xi}} = \tilde{\boldsymbol{\omega}}^* \cdot \boldsymbol{\xi}, \\ \boldsymbol{\xi}(0) = \mathbf{r}_0 - \mathbf{j} \frac{\mathbf{v}_0 + \tilde{\boldsymbol{\omega}}_0 \cdot \mathbf{r}_0}{\omega_*}, \end{cases}$$

where

$$(16) \quad \tilde{\boldsymbol{\omega}}^*(t) = -\tilde{\boldsymbol{\omega}}(t) + \mathbf{j} \omega_* \mathbf{I}_3, \quad t \geq 0,$$

with $\tilde{\boldsymbol{\omega}}_0 = \tilde{\boldsymbol{\omega}}(0)$ and $\mathbf{I}_3$ the identity matrix of order 3. From (11) it follows that the matrix $\tilde{\boldsymbol{\omega}}^*$ defined by (16) satisfies the condition

$$(17) \quad \tilde{\boldsymbol{\omega}}^*(t_1) \cdot \tilde{\boldsymbol{\omega}}^*(t_2) = \tilde{\boldsymbol{\omega}}^*(t_2) \cdot \tilde{\boldsymbol{\omega}}^*(t_1), \quad \forall t_1, t_2 \geq 0,$$

so that the solution to the problem (15) is given by

$$(18) \quad \xi(t) = \exp \left(\int_0^t \tilde{\omega}^*(\tau) d\tau \right) \cdot \xi(0), \quad t \geq 0.$$

From (14) it follows that the solution to the problem (13) represents the real part of (18), which is

$$(19) \quad \begin{aligned} \mathbf{r}(t) = \operatorname{Re} \xi(t) &= \operatorname{Re} \left(\exp \left(\int_0^t \tilde{\omega}^*(\tau) d\tau \right) \right) \cdot \operatorname{Re} \xi(0) - \\ &\quad - \operatorname{Im} \left(\exp \left(\int_0^t \tilde{\omega}^*(\tau) d\tau \right) \right) \cdot \operatorname{Im} \xi(0), \end{aligned}$$

where

$$(20) \quad \begin{aligned} \exp \left(\int_0^t \tilde{\omega}^*(\tau) d\tau \right) &= \exp \left(\int_0^t (-\tilde{\omega}(\tau) + j \omega_* \mathbf{I}_3) d\tau \right) = \\ &= e^{j \omega_* \mathbf{I}_3 t} \cdot \exp \left(- \int_0^t \tilde{\omega}(\tau) d\tau \right) = \\ &= (\cos \omega_* t + j \sin \omega_* t) \exp \left(- \int_0^t \tilde{\omega}(\tau) d\tau \right). \end{aligned}$$

The relations (19) and (20) lead us to the solution to the problem (13) expressed by

$$(21) \quad \begin{aligned} \mathbf{r}(t) &= \cos \omega_* t \exp \left(- \int_0^t \tilde{\omega}(\tau) d\tau \right) \cdot \mathbf{r}_0 + \\ &\quad + \sin \omega_* t \exp \left(- \int_0^t \tilde{\omega}(\tau) d\tau \right) \cdot \frac{\mathbf{v}_0 + \tilde{\omega}_0 \cdot \mathbf{r}_0}{\omega_*}, \end{aligned}$$

or the equivalent expression

$$(22) \quad \mathbf{r}(t) = \exp \left(- \int_0^t \tilde{\omega}(\tau) d\tau \right) \cdot \left(\cos \omega_* t \mathbf{r}_0 + \sin \omega_* t \frac{\mathbf{v}_0 + \tilde{\omega}_0 \cdot \mathbf{r}_0}{\omega_*} \right).$$

Considering the column matrix function

$$(23) \quad \psi_0(t) = \cos \omega_* t \mathbf{r}_0 + \sin \omega_* t \frac{\mathbf{v}_0 + \tilde{\omega}_0 \cdot \mathbf{r}_0}{\omega_*}, \quad t \geq 0,$$

the solution (22) becomes

$$(24) \quad \mathbf{r}(t) = \exp \left(- \int_0^t \tilde{\omega}(\tau) d\tau \right) \cdot \psi_0(t), \quad t \geq 0.$$

As proven in the appendix, the expression of the exponential involved in relation (24) is given by

$$(25) \quad \exp \left(- \int_0^t \tilde{\omega}(\tau) d\tau \right) = \mathbf{I}_3 - \sin \alpha(t) \tilde{\mathbf{u}} + (1 - \cos \alpha(t)) \tilde{\mathbf{u}}^2,$$

where

$$(26) \quad \alpha(t) = \int_0^t \omega(\tau) d\tau, \quad t \geq 0$$

and $\tilde{\mathbf{u}}$ is the antisymmetric matrix corresponding to the constant vector $\mathbf{u}$ that specifies the fixed direction of the angular velocity vector $\boldsymbol{\omega}$.

Therefore the solution to the Cauchy problem (13) is

$$(27) \quad \mathbf{r}(t) = [\mathbf{I}_3 - \sin \alpha(t) \tilde{\mathbf{u}} + (1 - \cos \alpha(t)) \tilde{\mathbf{u}}^2] \cdot \boldsymbol{\psi}_0(t).$$

Applying the correspondence Λ^{*-1} to relation (27) and taking into account (8), we obtain the vectorial solution to problem (12)

$$(28) \quad \mathbf{r} = \boldsymbol{\psi}_0 - \sin \alpha(t) \mathbf{u} \times \boldsymbol{\psi}_0 + (1 - \cos \alpha(t)) \mathbf{u} \times (\mathbf{u} \times \boldsymbol{\psi}_0),$$

where

$$(29) \quad \boldsymbol{\psi}_0(t) = \Lambda^{*-1} [\boldsymbol{\psi}_0(t)] = \cos \omega_* t \mathbf{r}_0 + \sin \omega_* t \frac{\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0}{\omega_*}.$$

After elementary transformations and taking into account that $\mathbf{u} = \boldsymbol{\omega}(t)/\omega(t)$, the relation (28) can be written in the equivalent form

$$(30) \quad \mathbf{r} = \frac{\boldsymbol{\omega} \cdot \boldsymbol{\psi}_0}{\omega^2} \boldsymbol{\omega} - \sin \alpha(t) \frac{\boldsymbol{\omega} \cdot \boldsymbol{\psi}_0}{\omega} \boldsymbol{\omega} - \cos \alpha(t) \frac{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\psi}_0)}{\omega^2},$$

which represents the exact solution to the Cauchy problem (12).

This result can be expressed by the following

Theorem 1. *If $\boldsymbol{\omega} : [0, +\infty) \rightarrow \mathcal{V}_3$ is a derivable vectorial function of fixed direction, and $\omega_* \neq 0$, then the solution to the Cauchy problem*

$$(31) \quad \begin{cases} \ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \omega_*^2 \mathbf{r} = \mathbf{0}, \\ \mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(0) = \mathbf{v}_0 \end{cases}$$

is represented by

$$(32) \quad \mathbf{r} = \frac{\boldsymbol{\omega} \cdot \boldsymbol{\psi}_0}{\omega^2} \boldsymbol{\omega} - \sin \alpha(t) \frac{\boldsymbol{\omega} \times \boldsymbol{\psi}_0}{\omega} - \cos \alpha(t) \frac{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\psi}_0)}{\omega^2},$$

where

$$(33) \quad \boldsymbol{\psi}_0 = \cos \omega_* t \mathbf{r}_0 + \sin \omega_* t \frac{\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0}{\omega_*}, \quad \boldsymbol{\omega}_0 = \boldsymbol{\omega}(0),$$

with

$$(34) \quad \alpha(t) = \int_0^t \omega(\tau) d\tau, \quad t \geq 0$$

and $\omega(t) = \boldsymbol{\omega}(t) \cdot \mathbf{u}$, where $\mathbf{u}$ is a vector, with $|\mathbf{u}| = 1$, that defines the fixed direction of the angular velocity vector $\boldsymbol{\omega}$.

Remarks. 1. The function $\boldsymbol{\psi}_0$ defined by (25) is the solution to the Cauchy problem

$$(35) \quad \begin{cases} \ddot{\mathbf{r}} + \omega_*^2 \mathbf{r} = \mathbf{0}, \\ \mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(0) = \mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0, \end{cases}$$

derived from (12) for $\boldsymbol{\omega} = \mathbf{0}$.

2. Solution (30) represents the vectorial form of the matrix expression (27). This has a special significance. Let

$$(36) \quad \mathbf{Q}(t) = \exp \left(- \int_0^t \tilde{\boldsymbol{\omega}}(\tau) d\tau \right) = \exp \left(- \int_0^t \boldsymbol{\omega}(\tau) d\tau \tilde{\mathbf{u}} \right) = \exp(-\alpha(t) \tilde{\mathbf{u}}),$$

$$t \geq 0,$$

with $\alpha(t)$ given by (26). Considering this notation, relation (27) can be expressed as

$$(37) \quad \mathbf{r}(t) = \mathbf{Q}(t) \cdot \boldsymbol{\psi}_0(t).$$

Matrix $\mathbf{Q}(t)$ defined by (36) is an orthogonal matrix for every $t \geq 0$. Indeed, since matrix $\tilde{\mathbf{u}}$ is antisymmetric ($\tilde{\mathbf{u}}^T = -\tilde{\mathbf{u}}$) then

$$(38) \quad \begin{aligned} \mathbf{Q} \cdot \mathbf{Q}^t &= \exp(-\alpha(t) \tilde{\mathbf{u}}) \cdot \exp(-\alpha(t) \tilde{\mathbf{u}}^T) = \\ &= \exp(-\alpha(t) \tilde{\mathbf{u}}) \cdot \exp(\alpha(t) \tilde{\mathbf{u}}) = \mathbf{I}_3. \end{aligned}$$

From (36) it follows that

$$(39) \quad \det \mathbf{Q}^T = \exp(-\alpha(t) \cdot \text{trace } \tilde{\mathbf{u}}) = e^0 = 1, \quad \forall t \geq 0.$$

The transformation defined by (33) represents a turn-tensor [7]. The angular velocity corresponding to this rotation is

$$(40) \quad \begin{aligned} \boldsymbol{\omega}^* &= \text{vect } \dot{\mathbf{Q}} \cdot \mathbf{Q}^T = \text{vect} (-\dot{\alpha}(t) \tilde{\mathbf{u}} \cdot \exp(\alpha(t) \tilde{\mathbf{u}}) \cdot \exp(\alpha(t) \tilde{\mathbf{u}})) = \\ &= \text{vect}(-\omega(t) \tilde{\mathbf{u}}) = \text{vect}(-\tilde{\boldsymbol{\omega}}) = -\boldsymbol{\omega}. \end{aligned}$$

Therefore transformation defined by (37) represents a turn-tensor with the angular velocity equal to $-\boldsymbol{\omega}$. Denote by $\mathcal{F}_{-\boldsymbol{\omega}}$ the tensorial operator for which the matrix representation is given by (37) (equivalent to (27)). The solution to the Cauchy problem (12) can be expressed as

$$(41) \quad \mathbf{r} = \mathcal{F}_{-\boldsymbol{\omega}}(\boldsymbol{\psi}_0).$$

Thus, Theorem 1 can be reformulated as

Theorem 2. *If $\boldsymbol{\omega} : [0, +\infty) \rightarrow \mathcal{V}_3$ is a derivable vectorial function of fixed direction, and $\omega_* \neq 0$, then the solution to the Cauchy problem*

$$(42) \quad \begin{cases} \ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \omega_*^2 \mathbf{r} = \mathbf{0}, \\ \mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(0) = \mathbf{v}_0 \end{cases}$$

can be obtained by applying the tensorial operator of rotation with angular velocity $-\boldsymbol{\omega}$

$$(43) \quad \mathcal{F}_{-\boldsymbol{\omega}} : \mathcal{V}_3^* \rightarrow \mathcal{V}_3^*,$$

$$\mathcal{F}_{-\boldsymbol{\omega}}(\cdot) = \frac{\boldsymbol{\omega} \cdot (\cdot)}{\omega^2} \boldsymbol{\omega} - \sin \alpha(t) \frac{\boldsymbol{\omega} \times (\cdot)}{\omega} - \cos \alpha(t) \frac{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\cdot))}{\omega^2},$$

where

$$(44) \quad \alpha(t) = \int_0^t \omega(\tau) d\tau, \quad t \geq 0$$

and $\omega(t) = \boldsymbol{\omega}(t) \cdot \mathbf{u}$, with $\mathbf{u}$ a vector ($|\mathbf{u}| = 1$) that defines the fixed direction of the angular velocity vector $\boldsymbol{\omega}$, to the solution of the Cauchy problem

$$(45) \quad \begin{cases} \ddot{\mathbf{r}} + \omega_*^2 \mathbf{r} = \mathbf{0}, \\ \mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(0) = \mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0, \end{cases}$$

where $\boldsymbol{\omega}_0 = \boldsymbol{\omega}(0)$.

The hodograph of the solution to problem (45), the vectorial function $\boldsymbol{\psi}_0$ defined by (33), is an ellipse, possibly degenerated, which has as conjugated diameters the directions specified by the vectors $\mathbf{r}_0$ and $\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0$. It follows that the hodograph of the solution to problem (12) can be visualised through the rotation with the angular velocity $-\boldsymbol{\omega}(t)$ of the plane of this ellipse.

In general, this is not a plane curve. The trajectory of the particle submitted to this motion is a plane curve if and only if the fixed direction of the angular velocity vector $\boldsymbol{\omega}$ is orthogonal to the plane determined by the vectors $\mathbf{r}_0$ and $\mathbf{v}_0$. Even in this particular case, the trajectory is not always a closed curve.

The behaviour of the solution (42) is sometimes astonishing. If, for example, the initial conditions of problem (45) are chosen so that the hodograph of the solution to the problem (45) is a circle, and the angular velocity vector $\boldsymbol{\omega}$ is constant and included into the plane determined by the vectors $\mathbf{r}_0$ and $\mathbf{v}_0$, then the trajectory of the particle covers densely the sphere of radius $|\mathbf{r}_0|$ if $\omega/\omega_* \in \mathbb{R} \setminus \mathbb{Q}$ (the two numbers are immeasurable) [1].

The hodograph of the solution to problem (45) is a circle if the initial conditions satisfy

$$(46) \quad \begin{cases} \mathbf{r}_0 \cdot \mathbf{v}_0 = 0, \\ r_0^2 - \frac{(\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0)^2}{\omega_*^2} = 0. \end{cases}$$

3. In case the angular velocity is constant, $\boldsymbol{\omega}(t) = \boldsymbol{\omega}$, with $\omega = |\boldsymbol{\omega}|$, solution (30) becomes

$$(47) \quad \mathbf{r}(t) = \frac{\boldsymbol{\omega} \cdot \boldsymbol{\psi}_0}{\omega^2} \boldsymbol{\omega} - \sin \omega t \frac{\boldsymbol{\omega} \times \boldsymbol{\psi}_0}{\omega^2} - \cos \omega t \frac{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\psi}_0)}{\omega^2},$$

where

$$(48) \quad \boldsymbol{\psi}_0 = \cos \omega_* t \mathbf{r}_0 + \sin \omega_* t \frac{\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_0}{\omega_*},$$

solution reported in [3] and [4].

4. In most of the classical mechanics treatises the mathematical model of the Foucault's pendulum is presented in the form of the reduced problem [1], [6], which is

$$(49) \quad \begin{cases} \ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \omega_0^2 \mathbf{r} = \mathbf{0}, \\ \mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(0) = \mathbf{v}_0, \end{cases}$$

where the angular velocity vector is constant, $\boldsymbol{\omega}(t) = \boldsymbol{\omega}$, and $\omega_0 \in \mathbb{R} \setminus \{0\}$.

A vectorial solution to this problem is presented in [2]. Paradoxically, the exact solution to the reduced problem (49) is harder to obtain than the one for the complete problem (12).

The solution to the Cauchy problem (49) is represented by

$$(50) \quad \mathbf{r} = \frac{\boldsymbol{\omega} \cdot \boldsymbol{\psi}_0}{\omega^2} \boldsymbol{\omega} - \sin \omega t \frac{\boldsymbol{\omega} \times \boldsymbol{\psi}_*}{\omega} - \cos \omega t \frac{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\psi}_*)}{\omega^2},$$

where $\omega = |\boldsymbol{\omega}|$ and

$$(51) \quad \boldsymbol{\psi}_0(t) = \cos \omega_0 t \mathbf{r}_0 + \sin \omega_0 t \frac{\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_0}{\omega_0},$$

$$\boldsymbol{\psi}_*(t) = \cos \omega_* t \mathbf{r} + \sin \omega_* t \frac{\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_0}{\omega_*}, \quad t \geq 0,$$

with $\omega_* = \sqrt{\omega^2 + \omega_0^2}$.

Appendix

Consider the vectorial function of fixed direction $\boldsymbol{\omega}(t) = \omega(t)\mathbf{u}$ with $|\mathbf{u}| = 1$, $\dot{\mathbf{u}} = \mathbf{0}$ and $\omega : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ a derivable real function. The corresponding antisymmetric matrix-function $\tilde{\omega}(t)$ has the form

$$(52) \quad \tilde{\omega}(t) = \omega(t)\tilde{\mathbf{u}},$$

where $\tilde{\mathbf{u}}$ is the antisymmetric matrix corresponding to vector $\mathbf{u}$. The Cayley-Hamilton theorem states that the matrix $\tilde{\mathbf{u}}$ satisfies its characteristic equation, that is

$$(53) \quad \tilde{\mathbf{u}}^3 + \tilde{\mathbf{u}} = \mathbf{0}.$$

The exponential of the antisymmetric matrix $\tilde{\omega}$ involved in (25) is determined using one of the well known methods [5]. Let

$$\begin{aligned} Q(t) &= \exp \left(- \int_0^t \tilde{\omega}(\tau) d\tau \right) = \exp \left(- \int_0^t \omega(\tau) d\tau \tilde{u} \right) = \\ (54) \quad &= I_3 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \alpha^k(t) \tilde{u}^k, \end{aligned}$$

with

$$(55) \quad \alpha(t) = \int_0^t \omega(\tau) d\tau, \quad t \geq 0.$$

Introducing the relation (53) into (54), the matrix $Q(t)$ is obtained into the form

$$\begin{aligned} (56) \quad Q(t) &= I_3 - \left[\frac{\alpha(t)}{1!} - \frac{\alpha^3(t)}{3!} + \frac{\alpha^5(t)}{5!} - \frac{\alpha^7(t)}{7!} + \dots \right] \tilde{u} + \\ &+ \left[\frac{\alpha^2(t)}{2!} - \frac{\alpha^4(t)}{4!} + \frac{\alpha^6(t)}{6!} - \frac{\alpha^8(t)}{8!} + \dots \right] \tilde{u}^2. \end{aligned}$$

It is well known that

$$\begin{aligned} (57) \quad &\frac{\alpha(t)}{1!} - \frac{\alpha^3(t)}{3!} + \frac{\alpha^5(t)}{5!} - \frac{\alpha^7(t)}{7!} + \dots = \sin \alpha(t), \\ &\frac{\alpha^2(t)}{2!} - \frac{\alpha^4(t)}{4!} + \frac{\alpha^6(t)}{6!} - \frac{\alpha^8(t)}{8!} + \dots = 1 - \cos \alpha(t). \end{aligned}$$

Whence the exponential matrix $Q(t)$ is given by

$$(58) \quad Q(t) = \exp \left(- \int_0^t \tilde{\omega}(\tau) d\tau \right) = I_3 - \sin \alpha(t) \tilde{u} + (1 - \cos \alpha(t)) \tilde{u}^2.$$

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REFERENCES

1. Arnold V., *Les méthodes mathématiques de la mécanique classique*. Éd. Mir, Moscou, 1976.
2. Condurache D., Braier A., *The Direct Study of Some Classical Problems of Theoretical Mechanics*. Bul. Inst. Polit. Iași, **XLI (XLV)**, 1-2, s. I (1995).
3. Condurache D., *New Symbolic Methods Into the Study of Dynamic Systems* (in Romanian). Master's Degree Thesis, Technical University "Gh. Asachi", Jassy, 1995.
4. Condurache D., *Symbolic Representation. Applications to Signal Theory and the Study of Dynamic Systems* (in Romanian). Ed. Nord-Est, Iași, 1996.
5. Corduneanu A., *Differential Equations - Applications to Electrotechnics* (in Romanian). Ed. Facla, Timișoara, 1981.
6. Landau L., Lifchitz E., *Mécanique*. Ed. Mir, Moscou, 1981.
7. Zhilin P. A., *Analysis of Free Rotations of Rigid Bodies*. ZAMM, **76** (1996).

O SOLUȚIE EXACTĂ A PROBLEMEI PENDULULUI FOUCAULT

(Rezumat)

Se prezintă o soluție exactă a problemei pendulului lui Foucault care utilizează o reprezentare simbolică a funcțiilor vectoriale prin matrici.

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AN APPROACH TO ROBUST FAULT DETECTION

I. A METHOD FOR DESIGNING FAULT DETECTION OBSERVERS

BY

MIHAELA-HANAKO MATCOVSCHI

Abstract. This paper presents a method for the design of unknown input fault detection observers. In case that the mathematical requirements are accomplished, this method provides a perfect decoupling between unknown inputs and faults. If this is not possible because certain prerequisites are not fulfilled, an optimal compromise is described. This design is based on the Observer Canonical Form of the system. Finally, a Matlab package based on the presented theoretical approach is described.

Key words: dynamic systems, safety and reliability, failure detection and isolation, robustness.

1. Introduction

As a response to an increasing demand in process safety and reliability, many failure detection and isolation (FDI) methods, based on dynamic models and the inherent analytical redundancy appeared. The decisive significance of robustness in FDI was recognized and several ways to increase the robustness have been suggested. In mathematical terms, the robustness problem can be interpreted as a min-max problem, since the residual generated by the FDI algorithm should be minimally sensitive to unknown inputs and maximally sensitive to faults [2].

This paper starts with the mathematical formulation of the problem and theorems concerning achievable robustness following the work of W ü n n e n b e r g [4]. A method for the design of fault detection observers providing a perfect decoupling between unknown inputs and faults is described or, in case this is not possible, an optimal compromise is depicted. Based on the proposed method a Matlab package of functions is presented. The application results for an inverted pendulum are to be given in the second part of this paper.

2. Theoretical Approach

Consider a linear continuous or discrete time-invariant causal system given by the state-space description (Fig. 1):

$$(1) \quad \begin{aligned} \mathcal{D}I_n x(t) &= Ax(t) + Bu(t) + E_d d_d(t) + K_d f_d(t); \quad x(t_0) = x_0, \\ y(t) &= Cx(t) + Du(t) + E_m d_m(t) + K_m f_m(t). \end{aligned}$$

Here $x(t)$ represents the n -dimensional state vector, $u(t)$ the r -dimensional known input vector; $E_d d_d(t)$ characterizes a s -dimensional unknown input vector with known distribution matrix E_d acting directly onto the system dynamics equation (1); $K_d f_d(t)$ characterizes a l -dimensional unknown fault vector $f_d(t)$, such as actuator or component faults, with known distribution matrix K_d ; the m -dimensional measurement vector $y(t)$ is assumed to be available for further treatment; $E_m d_m(t)$ describes the disturbances acting as unknown inputs, where $d_m(t)$ is a s^* -dimensional unknown input vector with known distribution matrix E_m acting on system measurements; $K_m f_m(t)$ refers to the faults that corrupt the measurements, called instrument faults, with $f_m(t)$ a l^* -dimensional unknown fault event vector.

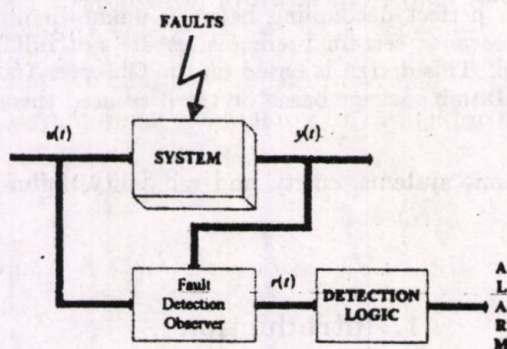


Fig. 1 — Dynamic system with unknown inputs and faults.

The signals $d_d(t)$ and $d_m(t)$, as well as $f_d(t)$ and $f_m(t)$ are assumed to be linearly independent. the matrices A , B , C , D , E_d , E_m , K_d and K_m (*nominal system matrices*) are real valued and of appropriate dimensions and I_n represents the n -dimensional unity matrix.

The operator $\mathcal{D}$ denotes either the differential operator $\mathcal{D}^k x(t) = d^k x(t)/dt^k$ for continuous time signals or, in the case of discrete time signals, the shift operator $\mathcal{D}^k x(t) = x(t + kT)$ with the constant sampling time T .

Definition 1 [4]. The system described by Eq. (1) is called *Unknown Input Fault Detectable* (UIFD) if there exists a t_0 with: $-\infty < t_0 < t_1$ so that, for all $x(t)$, $u(t)$, $d_d(t)$, $d_m(t)$ and for almost all $f_d(t) \neq 0$ and $f_m(t) \neq 0$ with $t_0 < t <$

$< t_1$, the knowledge of the vectors $u(t)$ and $y(t, t_0, u(t), d_d(t), d_m(t), f_d(t), f_m(t))$ for $t_0 \leq t \leq t_1$ implies $f_d(t) \neq 0$ or $f_m(t) \neq 0$ ¹).

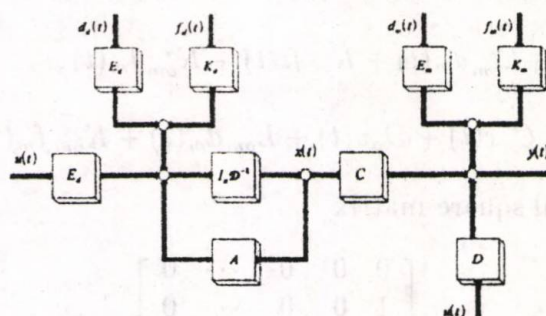


Fig. 2 — Unknown input fault detection observer.

Definition 2 [4]. The dynamic system (Fig. 2):

$$\begin{aligned} \mathcal{D}z(t) &= Fz(t) + Ju(t) + Gy(t), \\ (2) \quad r(t) &= L_1z(t) + L_2y(t), \end{aligned}$$

with $z(t)$ being the p -dimensional state of the system, $r(t)$ being the v -dimensional output signal, called residual, and $y(t)$ and $u(t)$ defined by Eqs. (1), is called an *Unknown Input Fault Detection Observer* (UIFDO) for the system (1) iff for any $u(t)$, $d_d(t)$ and $d_m(t)$:

- (i) for all x_0, z_0 , $\lim_{t \rightarrow \infty} r(t) = 0$ if $f_d(t) \neq 0$ and $f_m(t) \neq 0$;
- (ii) for a $t \times n$ -dimensional matrix T : $z_0 = Tx_0$ implies $z(t) = Tx(t)$ for all t and $f_d(t) = 0$ and $f_m(t) = 0$;
- (iii) for $z_0 = Tx_0$ and almost any arbitrary $f_d(t) \neq 0$ and $f_m(t) \neq 0$ follows $r(t) \neq 0$

holds.

The *Unknown Input Fault Detection Observer* (UIFDO) is a linear time-invariant and causal system that processes the available input and output data and generates a residual (Fig. 3). Based on the residual generated by this observer, the detection logic allows setting the alarm in case a fault is active in the system.

The key question, namely whether an UIFDO exists, is answered in Theorem 1, proven in 1990 by W ü n n e n b e r g [4], based on the Kronecker canonical form of the system.

Theorem 1. *The Unknown Input Fault Detection Observer for the system described by (1) exists iff the system is Unknown Input Fault Detectable.*

¹) In case of continuous time systems the definition implies that the knowledge of the input and the available output signals for an arbitrary small time interval allows the unique conclusion that a fault is present in the system. The term *almost all* expresses that some exceptions exist, some faults cannot be detected even if the system is UIFD.

Definition 3. The single-output *Observer Canonical Form* (OCF) of the dynamic system described by (1) is defined by the following system representation:

$$(3) \quad \begin{aligned} \mathcal{D}I_{so}x_o(t) &= F_o x(t) + G_o y(t) + B_o u(t) + E_{od}d_d(t) + \\ &+ E_{om}^* d_m(t) + K_{od}f_o(t) + K_{om}^* f_m(t), \\ y_o(t) &= C_o x(t) + D_o u(t) + E_{om} d_m(t) + K_{om} f_m(t), \end{aligned}$$

with F_o a so dimensional square matrix

$$(4) \quad F_o = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

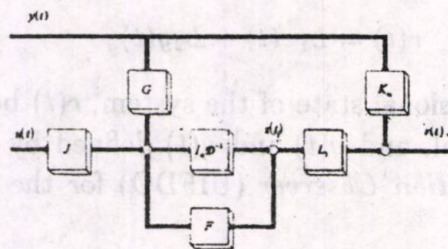


Fig. 3 — Fault detection observer.

The OCF measurement matrix C_o is a row vector of dimension so

$$(5) \quad C_o = (0, \dots, 0, 1),$$

the transformation matrix that describes the conversion of the dynamic system (1) to the OCF is defined as

$$(6) \quad x_o(t) = T_o x(t)$$

and the transformation of the output variables is defined as

$$(7) \quad y_o(t) = V_o y(t).$$

Imposing certain requirements to the transforming matrices T_o and V_o allows a unique derivation of the transforming equations, as stated in

Lemma 1. The transformation of a linear system described by (1) is defined by

$$(8) \quad x_o(t) = T_o x(t),$$

L e m m a 1. *The transformation of a linear system described by (1) is defined by*

$$(8) \quad x_o(t) = T_o x(t),$$

with

$$(9) \quad T_o = \begin{bmatrix} v_{so}^T C A^{so-1} + v_{so-1}^T C A^{so-2} + \dots + v_1^T C \\ v_{so}^T C A^{so-2} + \dots + v_2^T C \\ \vdots \\ v_{so}^T C A + v_{so-1}^T C \\ v_{so}^T C \end{bmatrix},$$

where

$$(10) \quad v_o^T = [v_0^T, v_1^T, \dots, v_{so}^T]$$

is a vector chosen as an annihilator to

$$(11) \quad Q = [C^T \ A^T C^T \ A^{2T} C^{2T} \ \dots \ A^{soT} C^T]^T.$$

which means that

$$(12) \quad v_o^T Q = 0.$$

The remaining matrices are given by

$$(13) \quad G_o = \begin{bmatrix} v_0^T \\ v_1^T \\ v_2^T \\ \vdots \\ v_{so-1}^T \end{bmatrix}, \quad \begin{aligned} B_o &= T_o B, \\ K_{od} &= T_o K_d, \\ K_{om}^* &= -G_o K_m, \\ D_o &= V_o D, \\ K_{om} &= V_o K_m, \end{aligned}$$

with

$$(14) \quad V_o = v_{so}^T.$$

The matrices E_{od} , E_{om}^* and E_{om} satisfy the conditions

$$(15) \quad E_{od} = T_o E_d = 0, \quad E_{om}^* = -G_o E_m = 0, \quad E_{om} = V_o E_m = 0.$$

Furthermore the output is transformed by

$$(16) \quad y_o(t) = V_o y(t).$$

The left null space of the matrix Q is known as the *Parity Space*. Based on the OCF that is also understood as a partial description for the undisturbed portion of the system, the UIFDO can be determined. The conditions for the existence of an UIFDO for a given system are presented in

Theorem 2. *The n -dimensional system described by (1) is Unknown Input Fault Detectable iff one variable s_o and one vector v_o^T can be found that fulfill the requirements*

$$(C1) \quad v_o^T Q = 0 \quad \text{and} \quad (C2) \quad v_o^T H_E = 0,$$

with

$$(17) \quad Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{s_o} \end{bmatrix} \quad \text{and} \quad H_E = \begin{bmatrix} E_m^{**} & 0 & 0 & \cdots & 0 \\ CE_d^{**} & E_m^{**} & 0 & \cdots & 0 \\ CAE_d^{**} & CE_d^{**} & E_m^{**} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ CA^{s_o-1}E_d^{**} & CA^{s_o-2}E_d^{**} & \cdots & \cdots & E_m^{**} \end{bmatrix},$$

where

$$(18) \quad E_m^{**} = [0_{m,s}, E_m], \quad E_d^{**} = [E_d, 0_{n,s}]$$

and $0_{m,s}$ a zero matrix of dimension $m \times s$ and $0_{n,s}$ a zero matrix of dimension $n \times s$.

Furthermore

$$(C3) \quad v_o^T H_{K,i} \neq 0$$

must hold for each $H_{K,i}$, ($i = 1, \dots, l + l^*$) with

$$(19a) \quad H_{K,i} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ CK_{d,i} & 0 & 0 & \cdots & 0 \\ CAK_{d,i} & CK_{d,i} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ CA^{s_o-1}K_{d,i} & CA^{s_o-2}K_{d,i} & \cdots & \cdots & 0 \end{bmatrix} \quad \text{for } i = 1, \dots, l,$$

where $K_{d,i}$ denotes the i -th column of the matrix K_d and

$$(19b) \quad H_{K,i+l} = \begin{bmatrix} K_{m,i} & 0 & 0 & \cdots & 0 \\ 0 & K_{m,i} & 0 & \cdots & 0 \\ 0 & 0 & K_{m,i} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & K_{m,i} \end{bmatrix} \quad \text{for } i = 1, \dots, l^*,$$

where $K_{m,i}$ denotes the i -th column of the matrix K_m .

The UIFDO design equivalents to the design of a full order Luenberger Observer for the part of the system that is given in the OCF. In that context, the system output signal $y(t)$ is interpreted as a known input to the partial description. The resulting UIFDO is described by

$$(20) \quad \begin{aligned} Dz(t) &= F_o z(t) + B_o u(t) + Gy(t) + Hr(t), \\ r(t) &= -C_o z(t) + V_o y(t). \end{aligned}$$

By comparison, the obtained observer with the UIFDO according to Eqs.(2), with $p = so$, the relations between the UIFDO matrices are

$$(21) \quad F = F_o - HC_o, \quad J = B_o, \quad G = G_o + HV_o, \quad L_1 = -C_o, \quad L_2 = V_o.$$

The matrix H is chosen so that the eigenvalues of matrix F are the ones chosen by the designer.

The computation of the UIFDO that completely suppresses the effect of the unknown inputs, while maintaining the sensitivity to almost all faults, shows that certain mathematical requirements for the system must be fulfilled. These requirements can often not be satisfied. The way to solve the problem in this case is to find an optimal solution by defining a *performance index* that considers both the unknown inputs and the faults simultaneously. The optimality is understood as a compromise in the sense that the effect of the unknown inputs onto the residual is minimized, while the effect of the faults onto the residual is maximized. In this case a so-called *Optimal Fault Detection Observer* (OFDO) can be determined.

Definition 4. The *optimal parity vector* v^T for the system described by (1) minimizes the performance index

$$(22) \quad J_p = \frac{\|v^T H_E\|_2}{\|v^T H_K\|_2},$$

under the constraint

$$(C1) \quad v_o^T Q = 0,$$

where matrices Q and H_2 are given by (19) and matrix H_K is determined by

$$(23) \quad H_K = \begin{bmatrix} K_m^{**} & 0 & 0 & \cdots & 0 \\ CK_d^{**} & K_m^{**} & 0 & \cdots & 0 \\ CAK_d^{**} & CK_d^{**} & K_m^{**} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ CA^{so-1}K_d^{**} & CA^{so-2}K_d^{**} & \cdots & \cdots & K_m^{**} \end{bmatrix},$$

with

$$(24) \quad K_m^{**} = [0_{m,l}, K_m], \quad K_d^{**} = [K_d, 0_{n,l^*}],$$

where $0_{m,l}$ a zero matrix of dimension $m \times l$ and $0_{n,l^*}$ a zero matrix of dimension $n \times l^*$.

In order to find the optimal parity vector v^T , the problem is reformulated. The vector v^T can be assumed to be

$$(25) \quad v^T = w^T V,$$

with V chosen as a basis for all solutions of the Eq. (C1). The performance index can be rewritten as

$$(26) \quad J_p = \frac{w^T V H_E H_E^T V^T w}{w^T V H_K H_K^T V^T w},$$

and the problem now is to determine the vector w^T that minimizes the index (28). The solution to the optimization problem can be found e.g. in Gantmacher [1] for regular matrix pencils and is stated in

Theorem 3. *The smallest characteristic value of the regular pencil*

$$(27) \quad (w^T V H_E H_E^T V^T w) - \lambda (w^T V H_K H_K^T V^T w)$$

is the minimum of the ratio of the forms $w^T V H_E H_E^T V^T w$ and $w^T V H_K H_K^T V^T w$,

$$(28) \quad \lambda_{\min} = \min \frac{w^T V H_E H_E^T V^T w}{w^T V H_K H_K^T V^T w}$$

and this minimum is only assumed for principal vectors of the characteristic value $\lambda_{\min}$.

The characteristic value λ_i and its corresponding principal vector w_i of the matrix pencil (29) are solutions to the problem

$$(30) \quad V H_E H_E^T V^T w_i = \lambda_i V H_K H_K^T V^T w_i,$$

which is also known in the literature as *the generalized eigenvalue-eigenvector problem*. Therefore the minimal eigenvalue is the optimal performance index and its corresponding eigenvector is the optimal selector for the parity vector v^T . In addition one can find in G a n t m a c h e r [1] that the characteristic values of the above pencil are real and the corresponding principal vectors are orthogonal to each other. Therefore, one can always find a solution.

3. The UIFDOOCF Matlab Package

We are now presenting a Matlab package of functions that can be used for the design of a scheme of observers for fault detection. This package is based on the design procedure of an Unknown Input Fault Detection Observer using the Observer Canonical Form.

The main function of the package is *uifdoocf* that, based on the nominal system matrices of the dynamic system described by (1), computes the matrices of the UIFDO described by (2). Syntax:

$$[F, J, G, L1, L2] = uifdoocf(A, B, C, D, Ed, Em, Kd, Km).$$

Therefore, the user must provide a system description according to Eqs. (1). From the previous presentation, the steps for the UIFDO computation can be summarized as follows:

Step 1: For the input matrices the dimensional consistency is checked by the function *dim_chk*. This function returns the empty matrix if they are dimensionally consistent, or an error message string if they are not. Also, the observer order s_o is initialized.

Step 2: Initialization of the auxiliary matrices E_m^{**} and E_d^{**} given by (18), and K_i^{**} and K_d^{**} from (24). Matrices Q , H_E and H_K are computed for the first iteration by the function *m_aux*.

Step 3: The basis of the parity space according to Eq. (C1) must be found, so that it must be a basis for the left null space of the matrix H_E simultaneously, according to Eq. (C2). This basis is computed by the function *int_null*. This computation is based on the following:

- (i) an orthonormal basis for the left null space of matrix Q is a basis for right null space of Q^T ;
- (ii) the intersection of the left null spaces for the matrices Q and H_E $\text{null}(Q) \cap \text{null}(H_E)$ is the basis for the right null space of the matrix $[Q \ H_E]^T$.

In case that the intersection of the left null spaces is void, the order of the parity space is increased. The new matrices Q and H_E that correspond to the new iteration are computed recursively by the function *m_new*.

Remark. Based on the correspondence existing between the Observer Canonical Form and the Kronecker Canonical Form of a system, as stressed out in W ü n e n b e r g [4], the upper limit of the parity space is $s_{\max} = \min(n - s, n - m + s^*)$. Therefore condition

$$(32) \quad s_0 \leq s_{\max} = \min(n - s, n - m + s^*)$$

is tested after every iteration.

Step 4: Out of the basis determined at Step 3*, a vector v^T that fulfills Eq. (C3) must be singled out. The *cond_3* function test (C3) for every vector of this basis. If (C3) is not fulfilled for any of the vectors of the basis determined at Step 3, the order so is increased. While condition (32) is fulfilled, Steps 3 and 4 are resumed, incrementing the order until a vector v^T satisfying the conditions (C1), (C2) and (C3) can be determined. The first order leading to a solution is held back.

Step 5: If an exact solution to Eqs. (C1), (C2) and (C3) has not been found, an *Optimal Fault Detection Observer* is designed using the *sol_opt* function. The order so is initialized with $so = 1$ and the basis of the parity space according to Eq. (C1) is found. In case that the parity space is void the order is increased. The first order leading to a solution is held back.

Step 6: The optimal parity vector v^T according to Theorem 3 is computed. Choosing V as a basis for all the solutions of Eq. (C1) matrices $\mathcal{A} = V H_E H_E^T V^T$ and $\mathcal{B} = V H_K H_K^T V^T$ are computed. Since pencil $\mathcal{A} - \lambda \mathcal{B}$ is symmetric-definite, all the eigenvalues are real. These are computed with the help of the Matlab function *eig*. The smallest characteristic value $\lambda_{\min}$ of the regular pencil $\mathcal{A} - \lambda \mathcal{B}$ is singled out, together with the corresponding eigenvector $w_{\min}$. The optimal parity vector v^T is computed with the help of (25). As pointed before, an optimal solution can always be found.

Remark. Employing this function, in case that $\lambda_{\min} = 0$, which is the minimum value of the performance index J_P , the optimal solution coincides with the exact solution and a perfect decoupling of the residuals from the unknown inputs is achieved.

Step 7: Based on the vector v^T determined at Step 4 (for the exact solution) or Step 6 (for the optimal solution), the Observer Canonical Form of the system has to be found according to Lemma 1. First of all the transformation matrix T_o is computed, according to (9), by the *m_transf* function. Using this matrix, the function *sist_ocf* determines the matrices of the OCF of the system and tests conditions (15). In case that these conditions are not fulfilled, an error message is generated.

Step 8: The matrices of the UIFDO are computed by the *m_uifdo* function based on the OCF of the system. Considering the matrices F_o and C_o determined at Step 7, the feedback matrix H must be determined so that the matrix of the observer given by $F = F_o - H C_o$ has the desired poles and the corresponding characteristic equation. The user introduces these poles, according to the purpose of the design. The Matlab function *place* is employed. Matrices J , G , L_1 and L_2 of the UIFDO

are computed according to (23). This concludes the UIFDO design, which is ready for implementation.

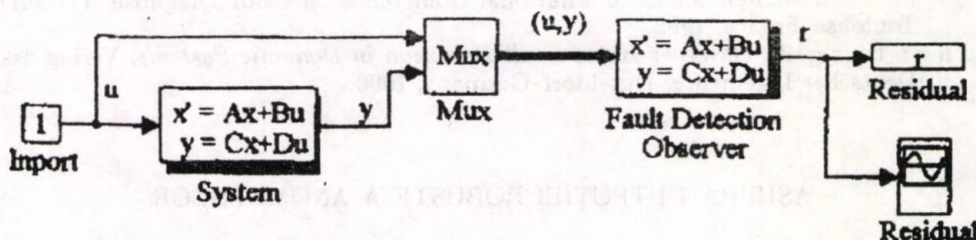


Fig. 4 — Use of an UIFDO in a Simulink model.

The package contains also an auxiliary function *sim_obs*, which allows the use of the designed UIFDO in a Simulink simulation model, as shown in Fig. 4. The matrices of the UIFDO are transformed in the form in which they can be used by a Simulink model the equations of the observer (2) being rewritten as

$$(31) \quad Dz(t) = Fz(t) + [J \ G] \cdot \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \quad r(t) = L_1 z(t) + [0 \ L_2] \cdot \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}.$$

This package has been successfully used for the study of the sensor faults for the Inverted Pendulum System. The results will be presented in the second part of this paper. The residuals obtained through simulation have proven the robustness of the observers designed using the *uifdoocf* function. A demo file showing these results has been realized. The residuals in the faulty cases plotted vs. the corresponding ones in the fault-free case have shown a perfect decoupling from the unknown inputs.

4. Conclusions

1. A method for the design of Unknown Input Fault Detection Observers based on the Observer Canonical Form of the system is presented.
2. The method provides a perfect decoupling between unknown inputs and faults. The UIFDO makes use of its robustness properties in order to decouple different faults and get a correct fault decision.
3. An optimal compromise, useful in case that the perfect decoupling is not possible, has been described.
4. Finally, a Matlab package implementing the presented theoretical approach is proposed.

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REFERENCES

1. Gantmacher F. R., *The Theory of Matrices*. Chelsea Publishing Company, USA, 1959.

2. Frank P. M., Köppen B., *Review of Optimal Solutions to the Robustness Problem in Observer-based Fault Detection*. Proc. Instn. Mech. Engrs., **207** (1993).
3. Köppen B., Frank P. M., *Application of Observer-Based Fault Detection Schemes to the Inverted Pendulum*. International Conference on Fault Diagnosis TOOLDIAG'93, Toulouse-France, 1993.
4. Wünnenberg J., *Observer-Based Fault Detection in Dynamic Systems*. Verlag des Vereins Deutscher Ingenieure, Düsseldorf-Germany, 1990.

ASUPRA DETECȚIEI ROBUSTE A ANOMALIILOR

I. O metodă de proiectare a observerilor detectori de erori

(Rezumat)

Se propune o metodă de proiectare a observerilor detectori de erori pentru sisteme dinamice cu intrări necunoscute (perturbații, zgomote) bazată pe forma canonică a observerilor și care asigură decuplarea perfectă între intrările necunoscute și semnalele de anomalie. Pe această bază se elaborează un algoritm și programul de calcul în Matlab, cu care în partea a doua a lucrării se studiază pendulul inversat.

DEPENDENCE OF THE EXCESS MOLAR VOLUME OF BINARY LIQUIDS ON ITS COMPOSITION

BY

M. HABA and C. G. HABA

Abstract. An explicit correlation between the excess molar volume of several binary liquid mixtures and the excess mole fractions of their components was established and tested.

Key words: binary liquid, molar volume, excess mole fraction.

1. Introduction

Thermodynamic excess quantities play an important role in studying and characterizing the real systems with two or more components [1]. Their importance consists in the fact that they can be expressed similarly to those which refer to ideal systems. Namely, they can be treated as additive quantities which describe the coligative properties of the difference system: real system-ideal system.

In the case of real binary systems an important role among the thermodynamic extensive parameters play the molar volume and the excess molar volume. The last is generally expressed by means of the excess partial molar volumes of components. But, there also is the possibility to express them in terms of the effective mole fractions (also named thermodynamic activities) or the excess mole fractions. Such a way is very useful in certain situations especially in emphasizing the role of different molecular interactions [2].

The purpose of the present paper is to emphasize in detail this possibility and analyze several of its consequences.

2. Theory

2.1. As known, in the case of an ideal binary mixture, at which the molecular interactions can be neglected, the molar volume is composed additively of the molar volumes of components. Consequently, we can write

$$(1) \quad V^{\text{ideal}} = x_I V_I + x_{II} V_{II}.$$

Here, x_i ($i = I$ or II) is the mole fraction of the i -th component; it is defined by

$$(2) \quad x_i = \frac{n_i}{n_I + n_{II}},$$

where n_i is the number of moles with that the i -th component takes part to forming the binary system.

In the case of real systems, characterized by molecular interactions, formula (1) becomes invalid; it must be replaced by one of the following formulas:

$$(3a) \quad V^{\text{real}} = x_I V'_I + x_{II} V'_{II},$$

or

$$(3b) \quad V^{\text{real}} = a_I V_I + a_{II} V_{II}.$$

Here

$$(4a) \quad V'_i = \frac{d}{dn_i} [(n_I + n_{II}) V^{\text{real}}]$$

is the partial molar volume of the i -th component and

$$(4b) \quad a_i = \gamma_i x_i$$

represents the effective mole fraction, i.e. the activity of the i -th component, while γ_i is a proportionality coefficient which is referred to as activity coefficient.

Using Eqs. (1) and (3 a-b) we can define the excess molar volume by one of the formulas:

$$(5a) \quad V^E \equiv V^{\text{real}} - V^{\text{ideal}} = x_I V_I^E + x_{II} V_{II}^E,$$

or

$$(5b) \quad V^E \equiv V^{\text{real}} - V^{\text{ideal}} = V_I x_I^E + V_{II} x_{II}^E.$$

Here, $V_i^E \equiv V'_i - V_i$ is the excess molar volume of the i -th component, and $x_i^E \equiv a_i - x_i$ is its excess mole fraction.

Taking into account that, by definition,

$$(6a - b) \quad a_I + a_{II} = 1 \quad \text{and} \quad x_I + x_{II} = 1$$

and using the notations $a \equiv a_{II}$ and $x \equiv x_{II}$, Eq. (5b) gets the form

$$(7) \quad V^E = (a - x)(V_{II} - V_I) = x^E \delta V,$$

where $x^E \equiv a - x$ represents the excess mole fraction of the second component (considered by us as being more active than the first one), and $\delta V \equiv V_{II} - V_I$ is the corresponding molar volume difference.

If we define the normalized excess molar volume by the following equation:

$$(8) \quad V^{nE} = \frac{V^E}{\delta V},$$

then Eq. (7) gets the form

$$(9) \quad V^{nE} = x^E.$$

Consequently, the calculation of the excess molar volume comes thus to calculation of the excess mole fraction.

2.2. In order to calculate the excess mole fraction we use Eqs. (4b), (6a) and (6b). Combining these equations we find that the activity ratio, defined by

$$(10a) \quad A \equiv \frac{a_{II}}{a_I} = \frac{a}{1-a},$$

can be expressed as a function of the mole ratio

$$(10b) \quad X \equiv \frac{x_{II}}{x_I} = \frac{x}{1-x},$$

by the relationship

$$(10c) \quad A = \Gamma X,$$

where $\Gamma = \gamma_{II}/\gamma_I$ is the ratio of the activity coefficients.

From Eqs. (10a)...(10c) we can derive further on the following equivalent formulas for the activity of the second component:

$$(11a) \quad a = \frac{\Gamma X}{1 + \Gamma X},$$

or

$$(11b) \quad a = \frac{x}{1 - K(1 - x)},$$

where $K \equiv 1 - 1/\Gamma = (\gamma_{II} - \gamma_I)/\gamma_I$ is a "screening" factor [2].

It follows that if we know the ratio of activity coefficients, then we can calculate the excess mole fraction and, hence, the excess molar volume. Indeed, introducing expression (11b) in Eq. (9), we obtain

$$(12) \quad V^{nE} = \frac{Kx(1-x)}{1 - K(1-x)}.$$

In Fig.1 is given the plot of function (12) for different constant and subunitary values of K -parameter. These curves can also be characterized by the abscise

$$(13) \quad x_e = \frac{1}{K} \left[1 - K \pm (1 - K)^{1/2} \right],$$

where the index e has the signification: V^{nE} =extreme, if $x = x_e$. It follows from Eq. (12) that if $K \ll 1$, then

$$(14) \quad V^{nE} \approx Kx(1-x).$$

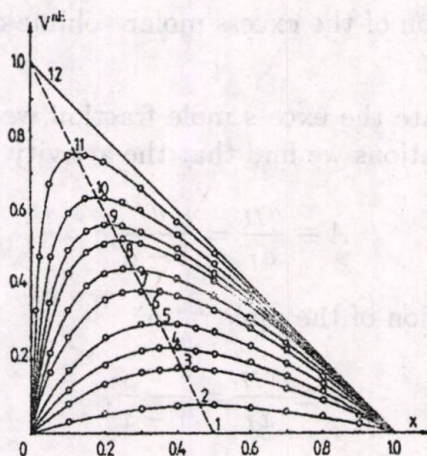


Fig. 1 — Dependence of the normalized excess molar volume on the liquid composition, when the liquid mixture is quasi-ideal. The given curves correspond to: 1 - $K=0$; 2 - $K=0.25$; 3 - $K=0.5$; 4 - $K=0.6$; 5 - $K=0.7$; 6 - $K=0.8$; 7 - $K=0.85$; 8 - $K=0.9$; 9 - $K=0.92$; 10 - $K=0.95$; 11 - $K=0.98$; 12 - $K=1$.

In this case, its plot is an arc of parabola having two zero points situated at $x = 0$ and $x = 1$, respectively, and a (positive or negative) peak situated at $x = 1/2$.

2.3. At present, there are numerous theoretical and empirical formulas which allow the calculating the activity coefficients [1]. The oldest and frequently used of these are the *M a r g u l e s'* formulas [3]. In accordance with these

$$(15) \quad \ln \Gamma = \sum_j (b_{2j}x_1^{j+1} - b_{1j}x_2^{j+1}),$$

where b_{1j} and b_{2j} are constants depending on the temperature, and $j = \overline{1, N}$ is the summing index. The number N of the sum terms will be chosen in dependence of the complexity degree of the system. For instance, if $b_{11} = b_{21} = b$ and $b_{1j} = b_{2j} = 0$ for $j > 1$ (regular binary mixture), then

$$(16) \quad \ln \Gamma = b(1 - 2x) = \frac{q(1 - 2x)}{kT},$$

where q is the heat of mixing and T is the temperature.

In these circumstances, the curve that gives V^{nE} vs. x gets the form shown in Fig. 2. The full line corresponds to $q > 0$, while the broken line give the case $q < 0$. In other words, when $q > 0$, then the addition of 1 mole of second component to the mixture $I + II$ leads first time to increasing and then to decreasing of excess molar volume.

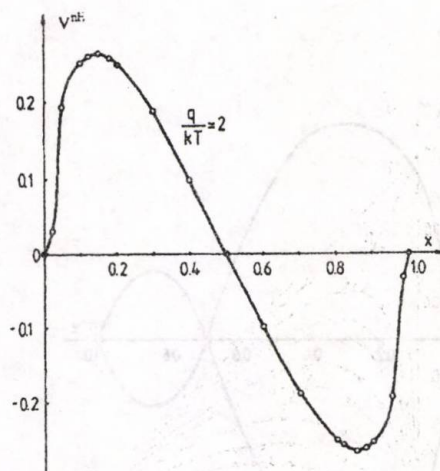


Fig. 2 — Dependence of the normalized excess molar volume on the liquid composition, when the liquid mixture is regular. It was plotted only the case when $q/kT = 2$.

2.4. A similar behaviour of the excess molar volume can be observed if we admit that the activity a is decomposable into two additive terms, a_1 and a_2 , every of they having a form given by Eq. (11b) and corresponding to a certain interaction process. In other words, we certain admit that

$$(17) \quad a = p_1 a_1 + p_2 a_2 = \frac{x p_1}{1 - K_1(1 - x)} + \frac{x p_2}{1 - K_2(1 - x)},$$

where K_1 and K_2 are the screening factors which correspond to the component processes, while p_1 and p_2 are their normalized weights, i.e. $p_1 + p_2 = 1$.

It is easy to see also in these conditions Eq. (17) can be transcribed as follows:

$$(18) \quad a = \frac{x}{1 - K_{eq}(1 - x)}.$$

But in this case the equivalent screening factor K_{eq} is now a function in x having the expression

$$(19) \quad K_{eq} = \frac{p_1 K_1 + p_2 K_2 - K_1 K_2(1 - x)}{1 - (p_1 K_2 + p_2 K_1)(1 - x)}.$$

Introducing this expression in Eq.(12), we obtain for the normalized excess molar volume the formula

$$(20) \quad V^{nE} = x(1-x) \frac{P(x - x_{inv})}{1 - S(1-x) + P(1-x)^2},$$

where

$$(21) \quad P = K_1 K_2, \quad S = K_1 + K_2 \quad \text{and} \quad x_{inv} = \frac{p_1}{K_2} + \frac{p_2}{K_1}.$$

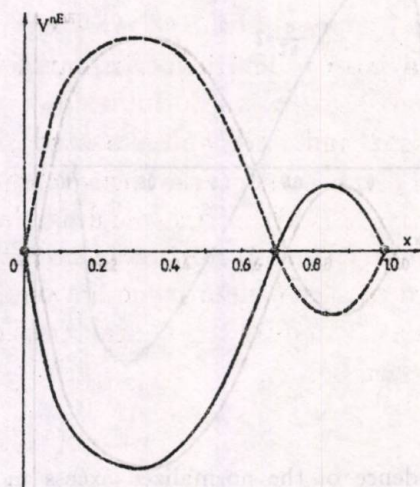


Fig. 3 — Normalized excess molar volume with three zero-points.

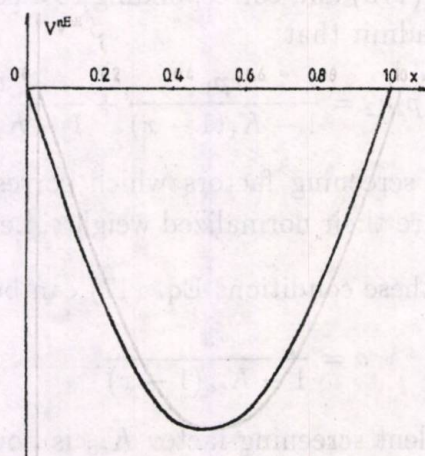


Fig. 4 — Normalized excess molar volume with two zero-points.

The plot of this function is illustrated in Fig. 3. It was chosen the situation when $x_{\text{inv}} \neq 0$. In this case, V^{nE} has three zero points. These points are situated at the abscises $x = 0$, $x = x_{\text{inv}}$ and $x = 1$, respectively. Between them there are at least a maximum and a minimum of the function V^{nE} . The abscises of these extremes, denoted by x_{max} and x_{min} , must satisfy one of the following inequalities:

$$(22) \quad 0 \leq x_{\text{min}} \leq x_{\text{inv}} \leq x_{\text{max}} \leq 1 \quad \text{or} \quad 0 \leq x_{\text{max}} \leq x_{\text{inv}} \leq x_{\text{min}} \leq 1.$$

It is interesting to observe that if the abscise of the inversion point do not satisfy inequalities (22), then one of the extremes of V^{nE} disappears and the corresponding plot gets the form given in Fig. 4.

3. Comparison with Experimental Data

In order to verify Eqs. (12) and (20) we have used the data referring to the excess molar volume of the following binary systems: cyclopentane+cyclooctane and cyclopentane+carbon tetrachloride. The corresponding data (published in [4], [5] and taken by us from [1], pp. 837-838) are plotted in Figs. 5 and 6. We see that, at least formal, there is a perfect concordance between theory and experiment. Consequently, if we use the data defining the characteristic points of these curves, then we can determine their parameters.

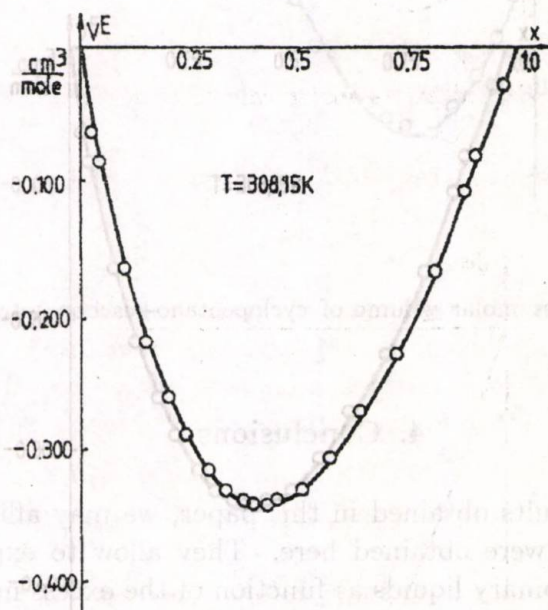


Fig. 5 — Excess molar volume of cyclopentane+cyclooctane.

So, for the mixture cyclopentane+cyclooctane, we have $x_{\text{min}} = 0.415$ and $V_{\text{min}}^{\text{nE}} = -0.34/38.97 = -0.00884$. Introducing these data in Eq. (12), we obtain for the

screening factor the value $K = -0.0128$. It follows that the given mixture is near ideal, i.e. its deviation degree Γ from the ideality is near 1. However, since $\Gamma = 0.9874 < 1$, it follows that the first mixture component is more active than the second one.

Cyclopentane+carbon tetrachloride is characterized by the following data: $x_{\min} = 0.3077$, $x_{\text{inv}} = 0.6923$, $x_{\max} = 0.8462$, $V_{\min}^{\text{nE}} = -0.005948$ and $V_{\max}^{\text{nE}} = 0.002379$. Solving Eq.(12) with these data, we find for the curve parameters the following values:

$$K_1 = -0.07712, \quad K_2 = -0.51807 \quad \text{and} \quad p_2 = 1 - p_1 = -0.20279.$$

These data indicate that the role of the second interaction process in the general variation of the excess molar volume is determinant, although its weight in Eq. (17) is smaller than that of the first component process.

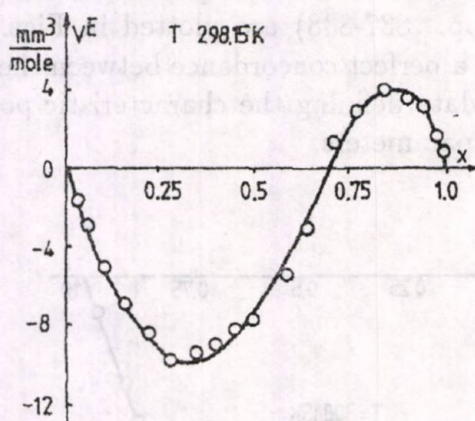


Fig. 6 — Excess molar volume of cyclopentano+carbon tetrachloride.

4. Conclusions

Summarizing the results obtained in this paper, we may affirm that a series of important relationships were obtained here. They allow to express explicitly the excess molar volume of binary liquids as function of the excess mole fractions of the components. They also permit to interpret the behaviour of excess molar volume of binary liquids in terms of molecular interactions.

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REFERENCES

1. Murgulescu I. G., Segal E., *Introduction to Physical Chemistry* (in Romanian). Vol. II.1, Ed. Acad. R.S.R., Bucharest, 1979.
2. Haba M., *Spectrosc. Lett.*, **19**, 505 (1986).
3. Margules M., *Sitzungsber. Akad. Wiss. Wien*, **2**, 104, 1243 (1985).
4. Marsh K. N., *International Data Series. Selected Data on Mixtures, Ser. A*, **12**, 128 (1975).
5. Benson G. C., *International Data Series. Selected Data on Mixtures, Ser. A*, **1**, 9 (1973).

DEPENDENȚA VOLUMULUI DE EXCES AL UNUI
LICHID BINAR DE COMPOZIȚIA AMESTECULUI

(Rezumat)

Scopul lucrării îl constituie evidențierea modalității de a trata, în cazul amestecurilor binare lichide, abaterea volumului molar real de la cel ideal, în strinsă corelație cu excesul fracțiilor molare ale componentelor.

Se stabilește și se verifică, pentru diverse amestecuri binare lichide, corelația între volumul molar de exces și excesul fracțiilor molare ale componentelor.

ON THE DIELECTRIC BEHAVIOUR OF THE POLYMETHYLMETHACRYLATE

BY

DORIN CONDURACHE, EUGEN OSADEȚ and EMIL LUCA

Abstract. The time dependence of the relative dielectric polarization induced by the heat exposure in electric fields of different intensities is studied for polymethylmethacrylate (PMA). The variation of the relative dielectric permittivity and the tangent of the loss angle with the measuring frequency is analyzed. At least two mechanisms are identified for the polarization and relaxation of the dipole momentum induced in the PMA. These two mechanisms are explained by the contribution of the polar groups entering the PMA structure, in the remanent superficial charges.

Key words: polymethylmethacrylate, remanent polarization, relaxation mechanism.

1. Introduction

The orientation polarization has been considered for a long time as the most important mechanism of the appearance of a remanent polarization in a dielectric including polar groups in its molecules. Recently, conflicting results appeared, for instance, the mechanism of dielectric polarization by the jump of the carrier between two centers [1], [2], jump that can sometimes occur in sequence [3]

The time stability of the induced superficial charges and the frequency dependence of the complex dielectric permittivity of the PMA can be explained to a great extent within the frame of this last polarization model.

2. The Preparation of the PMA Electrets and the Measurement of Their Properties

PMA discs with the diameter of 30 mm and the thickness of 4 mm have been mechanically cut. The samples were cleaned and kept for 4 h at 140°C. The samples showed no perceptible superficial charges after cooling. Part of these samples were kept as non-polarized witness samples.

Other samples were placed between two metallic electrodes, endowed with a guard ring, a constant electric field of the order of 10 kV/cm being applied between them. Simultaneously the heating system was turned on. The temperature and the polarization current intensity have been monitored all along the treatment. The characteristics of the treatment are presented in Fig. 1, but more series of samples have been prepared in parallel at different electric field intensities. Consequently, the superficial charge densities were different from one set of samples to another.

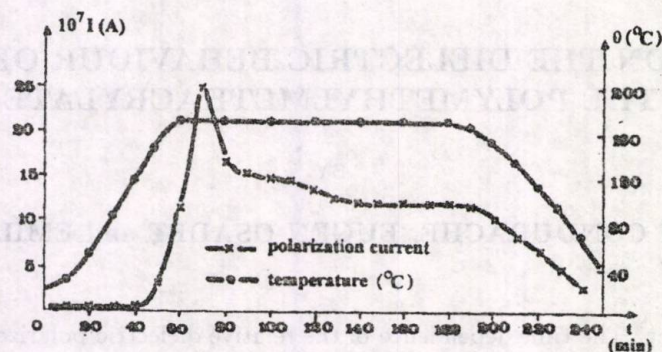


Fig. 1 — Polarization current intensity I and the temperature of thermal treatment θ vs. time t .

After the samples reached again the room temperature, the polarizing field was turned off. Later on, the samples were kept in dry atmosphere for two days.

The last group of samples was polarized for a short time (two minutes) in a field increasing up to 200 kV/cm at room temperature.

The remanent superficial charge has been measured for all the samples on their both faces for 60 days using the vibrating electrode method. The dielectric constant ϵ_r and the energy losses expressed by $\tan \delta$ have been also measured. These measurements have been performed by using a TESLA BM 311 Q- meter, at frequencies ranging between 8 kHz and 10 MHz and measuring fields of the order of 2.5 V/m.

3. Experimental Results and Discussion

One can notice from Fig. 1 that, at the beginning of the treatment, the current through the samples was very small (about 10^{-9} A). When the temperature reached 140°C, the polarization current quickly increases up to $24 \cdot 10^{-7}$ A. Even through both the temperature and the polarizing field were kept constant, the polarization current slowly decreases in time down to $12 \cdot 10^{-7}$ A, a value that remains constant as long as the temperature is held constant.

The descending segment of the curve $I = f(t)$ shows an anomaly after about $t = 10$ min. We assume that this segment represents in fact a second maximum, close

to the first one, less evident and partly hidden. These successive maxima of the polarization current during the formation treatment can be also found in other electrets with two polar groups in their molecules, such as sulpho-amino-methylphenoxiacetic acid [4]. The maxima positions and heights depends on the polar groups sites.

On the other hand, the relatively high value of the residual polarization current suggests a remarkable electric conductivity, explained by the charge migration toward the inhomogeneities.

Fig. 2 presents the time evolution of the superficial charge density in the polarized samples. The appearance of the heterocharges on the both sample faces has been noticed.

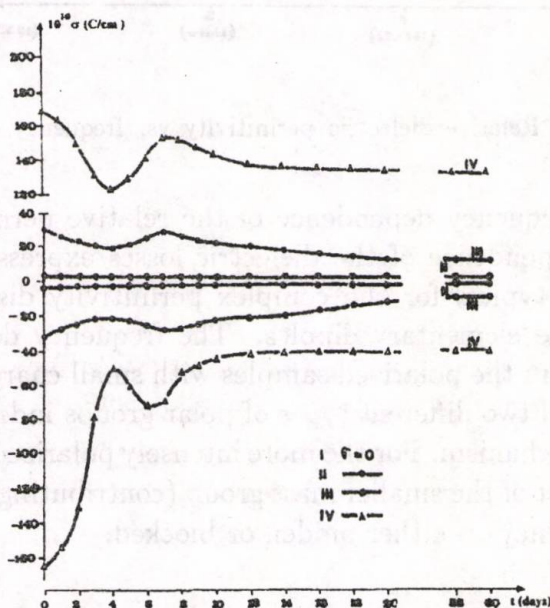


Fig. 2 — Superficial charge density in function of time t .

One can remark that in the slightly polarized ($\sigma_i = \pm 3 \cdot 10^{-10} \text{ C/cm}^2$) samples, the polarization decreases almost exponentially down to $0.5 \cdot 10^{-10} \text{ C/cm}^2$ after 10 days. The medium polarized samples ($\sigma_i = 3.5 \cdot 10^{-9} \text{ C/cm}^2$) show a superficial charge density decrease in two stages. The first stage begins just after the polarization and ends after two days. The second stage gets started after about 5 days from the initial polarization moment and ends in the tenth day. Later on, a slow decrease in time is seen, comparable with that found in the slightly polarized samples.

The samples polarized for a short time in a strong electric field and non-treated thermally present a more spectacular variation of the superficial charge density. Starting from practically equal initial values ($\sigma_i = \pm 16.6 \cdot 10^{-7} \text{ C/cm}^2$), during the fourth day a minimum is reached ($\sigma_4 = 10.5 \cdot 10^{-9} \text{ C/cm}^2$) on the positively charged face while a value of $-4.5 \cdot 10^{-9} \text{ C/cm}^2$ is reached on the negatively charged face;

after 7 days on the positive face σ almost returns to its initial value, while on the negative face it increases to $7.3 \cdot 10^{-9} \text{ C/cm}^2$. Later on, the superficial charge density decreases on the both faces getting stabilized from the twelfth day on at constant values of $13.3 \cdot 10^{-9} \text{ C/cm}^2$ and $-4.1 \cdot 10^{-9} \text{ C/cm}^2$ respectively.

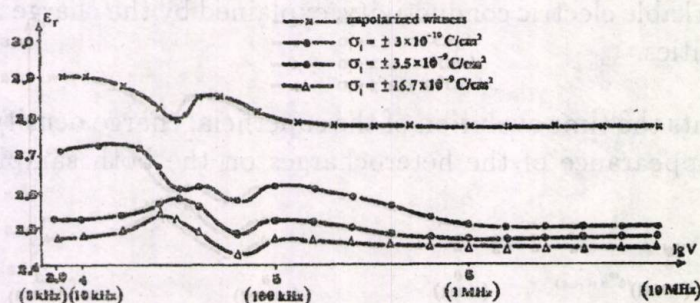


Fig. 3 — Relative dielectric permittivity vs. frequency.

Fig. 3 presents the frequency dependence of the relative permittivity, and Fig. 4 shows the frequency dependence of the dielectric losses expressed by $\tan \delta$. The aspect of these curves is typical for the complex permittivity dispersion near the resonance frequency of the elementary dipoles. The frequency dependence of the relative permittivity found in the polarized samples with small charges (Fig. 3, curve I) suggests the existence of two different types of polar groups independently taking part in the polarization mechanism. For the more intensely polarized samples (Fig. 3, curves III and IV) the effect of the smaller mass group (contributing in the dispersion maximum at higher frequency) is either hidden or blocked.

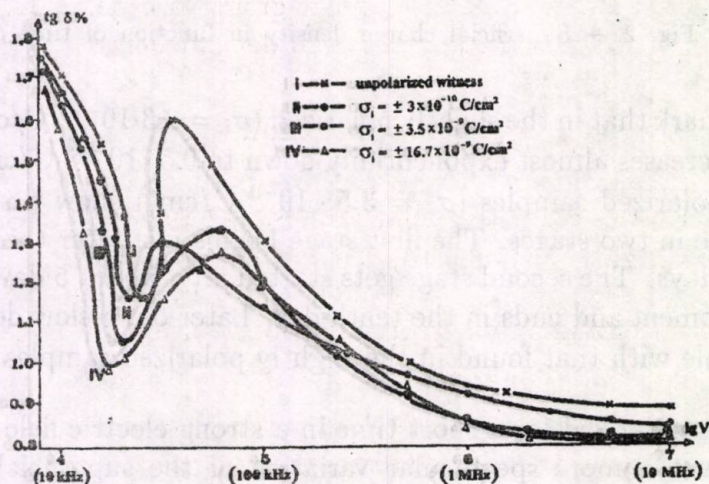


Fig. 4 — The tangent of the loss angle variation with the measuring frequency.

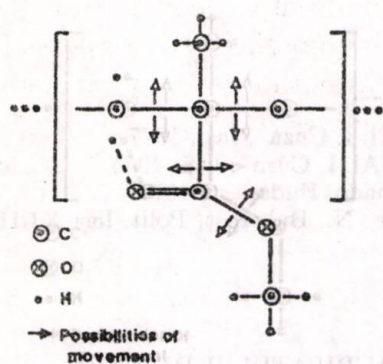


Fig. 5 — The chemical structure of PMA.

The positions and displacements of the extrema visible in Fig. 3, as well as the relaxations of the superficial electrization in two time stages, presented in Fig. 2, strengthen this assumption.

In fact, there are two polar groups in the PMA structure (Fig. 5): (a) the oxygen atom with double bond to the extra-chain carbon and (b) the oxygen atom linking the methyl radical to the same carbon atom.

The rigid double bond offers reduced possibilities for the orientation of the (a) group in the external field. The transversal oscillations of the chain (with a very large mass) correspond to a much lower resonance frequency [5], than that presented in Figs. 3 and 4.

The only shift that can be associated with the dipolar momentum modifications are those of the $-\text{COOCH}_3$ radical reported to the chain and those of the $-\text{OCH}_3$ group reported to the extra-chain carbon. The first shift can be blocked (at low energies) by an eventual hydrogen bond, indicated with dotted line in Fig. 5. Under this situation, the polar group (a) does not take part in the polarization phenomena.

At higher temperatures or frequencies or in a stronger field, the hydrogen bond is broken and over the oscillation of the $-\text{O}-\text{CH}_3$ group, the oscillation of the $-\text{COOCH}_3$ group overlaps. Since the mass of the latter group is double, the resonance frequency decreases almost by a factor of $\sqrt{2}$. The difference between the strength of the (C-O) interaction, characteristic to the oscillation of the $(-\text{OCH}_3)$ group, and that of the (C-C) interaction, characteristic to the oscillation of the $(-\text{COOCH}_3)$ group, combined with the difference between the masses of the two groups result in the difference noticed between the two resonance frequencies. For intense polarizations, the cumulated effects of the two polar groups are being favoured [6], which was confirmed by the experimentally observed dependencies.

4. Conclusion

1. In the dielectrics with two polar groups, at least two different mechanisms of polarization in electric field occur. These mechanisms can conceal each other under certain circumstances, yet they can be evidenced by the time evolution of the superficial electrization and by the frequency dependence of the dielectric permittivity and losses.

2. A quite strong polarization field (close to the breakdown voltage) applied even for a short time results in a long time remanent polarization of the dielectric, which diminishes the dielectric constant and losses.

REFERENCES

1. Scher H., Lax M., Phys. Rev, **87**, 4520 (1973).
2. Jonscher A. K., J. of Non-Cryst. Solids, **8 - 10**, 293 (1977).
3. Neagu E., Doctor Degree Thesis (in Romanian). Univ. "Al. I. Cuza", Iași, 1977.
4. Osadeț E., Doctor Degree Thesis (in Romanian). Univ. "Al. I. Cuza", Iași, 1985.
5. Hedwig P., *Dielectric Spectroscopy of Polymers*. Acad. Kiado, Budapest, 1977.
6. Osadeț E., Condurache D., Luca E., Foca N., Bul. Inst. Polit. Iași **XLII** (XLVI), 3 - 4, s. I, 117 (1996).

ASUPRA COMPORTĂRII DIELECTRICE A POLIMETACRILATULUI DE METIL

(Rezumat)

Se studiază dependența de timp a polarizării dielectrice remanente a polimetacrilatului de metil (PMA), indusă de o expunere la cald în câmpuri electrice de intensități diferite. Se analizează variația permitivității dielectrice relative și a tangentei unghiului de pierderi cu frecvența câmpului de măsură. Se identifică cel puțin două mecanisme de polarizare și de relaxare a momentului dipolar indus în PMA. Se explică aceste mecanisme prin contribuțiile grupărilor polare din structura PMA la sarcina superficială remanentă observată.

CARRIERS' MOBILITY WITHIN $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$ TYPE THIN FILMS

BY

ALEXANDRINA JEFLEA, EMIL LUCA and SOFIA MELINTE

Abstract. The obtaining of thermoelectric thin films that are competitive from the point of view of thermoelectric properties with the bulk material has led to observing several specific properties of the film and relate them to those of the bulk material. One of these parameters was the dependence of the carriers' mobility on temperature within thin film opposite to that within bulk material. $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$ was used as basic bulk material in obtaining thin films. The analyzed thin film has a thickness of $7.425\text{ }\mu\text{m}$, measured with an interference microscope.

Key words: thin film, thermoelectricity, transport phenomena.

1. Introduction

The obtaining of thermoelectric thin films that are competitive from the point of view of thermoelectric properties with the bulk material has led to observing several specific properties of the film and relate them to those of the bulk material. One of these parameters was the dependence of the carriers' mobility on temperature within thin film opposite to that within bulk material, $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$ being used as basic bulk material in obtaining thin films. The analyzed thin film has a thickness of $7.425\text{ }\mu\text{m}$, measured with an interference microscope.

2. Carriers' Concentration and Mobility within Bulk Material, at Various Temperatures

Taking notice of the difficulties encountered in the mechanical processing of Bi_2Te_3 bulk material and its alloys, we have used the Van der Pauw method in measuring the Hall effect. This method, as we all know, allows the determining of carriers' mobility and concentration for plane samples of irregular shape. The experimental conditions were those described in [1]. The Fig. 1 presents the assemblage used on this purpose.

The charge carriers' Hall mobility has, according to the Van der Pauw method, the expression [2]

$$(1) \quad \mu_H = \frac{d\Delta R_{BCDA}}{\rho B},$$

where ΔR_{BCDA} is the change of the fraction $R_{BCDA} = U_{AC}/I_{BD}$, caused by the presence of the magnetic field B ; d is the thickness of the sample; ρ is the electric resistivity of the sample; B is the induction of the applied magnetic field.

Carriers' concentration will be

Fig. 1 — The outline of the device used for measuring the Hall effect.

$$(2) \quad \rho = \frac{\mu_H}{\mu} \frac{1}{\rho \mu_H q} = \frac{r}{q \mu_H \rho},$$

where r is the Hall factor.

In deducing these formulas we supposed that all the contacts are placed strictly on the borders and have punctual dimensions. If these conditions are not fulfilled, we should introduce the C_H correction factors.

The calculus for carriers' mobility and concentration should pay attention to the fact that the spreading is made mainly on acoustic phonons, so $r = 1.17$.

We have obtained the following concentration values for the processed plates, at room temperature:

$$(3) \quad p = 6.82 \cdot 10^{25} \text{ m}^{-3}$$

and for mobility,

$$(4) \quad \mu = 1.403 \cdot 10^2 \text{ cm}^2/\text{Vs}.$$

One could get valuable information if we observe the dependence of the carriers' concentration and mobility on temperature. This concentration stays constant within the temperature range that we are interested in. The mobility will consequently present the same dependence as the conductivity. Experimental results are shown in Fig. 2.

Fig. 3 exposes the logarithmic dependence of resistivity and mobility as a function of temperature, and one can see that the slope is the same for both lines.

In case of the $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$ bulk material, at these temperatures the intrinsic conduction is still not excited, and the resistivity depends on the temperature only by

means of the mobility ($\mu \sim T^K$). From analyzing the acoustic spreading mechanism we have

$$(5) \quad \mu = \mu_{0,T} T^{-\frac{3}{2}}, \quad \mu_{0,T} \approx m^{*- \frac{5}{2}}.$$

Supposing the spreading mechanism is the same all over the temperature range we have studied, the deviation of k from the value of $3/2$ can be attributed to the dependence of $m \sim T^S$, that is [3]

$$(6) \quad S = \frac{2k - 3}{5}.$$

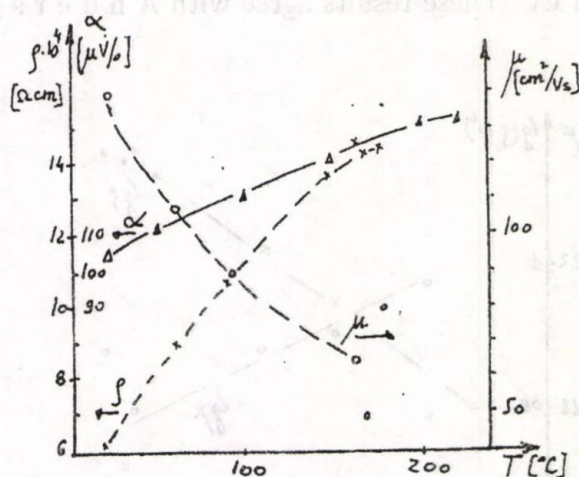


Fig. 2 — The dependence of resistivity, mobility and Seebeck coefficient on temperature for a $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$ bulk sample.

From experimental results we find that $k \in (1.77...2.18)$ and $s \in (0.18...0.24)$, in agreement with the literature. For the presented sample $k = 2.05$, $s = 0.22$.

All these measurements for bulk materials have been made in order to have a reference term for thin films and to check the measuring devices.

3. Hall Effect Measurements on Thin Films

Hall effect measurements on thin films required the making of a four probes measuring head; the probes were placed in the corners of a square. We permitted that the pressing force on a probe should be of ~ 0.3 N, according to [1]. In the calculus for specific values we have taken notice of the $C_H = 0.75$ correction factors, as a result of the device construction.

From the measurements done at room temperature we have concluded that mobility within thin films is lower than in bulk material, and the carriers' concentration is generally higher.

In the case of the ${}^3\text{K}_P^{14}$ sample, with $d = 7.452 \mu\text{m}$, the resistivity increases with the temperature, the mobility decreases, and the Seebeck coefficient has only small variations. Figs. 4 and 5 present the dependence of the resistivity, mobility and Seebeck coefficient as a function of temperature.

In opposition with bulk material, there appears, at this sample, a relationship between carriers' concentration and temperature. This relationship is shown in Fig. 6. We can evaluate the width of the forbidden area within the film as about 0.373 eV from processing experimental data under the form of $\ln(p/T^{3/2})$ as a function of $1/T$. On the other side, if we determine the change of the forbidden area due to intergranular barriers, using the P e t r i t z method [4] we find the height of the barriers $\phi = 0.0324$ eV. These results agree with A n d e r s o n's theory [5].

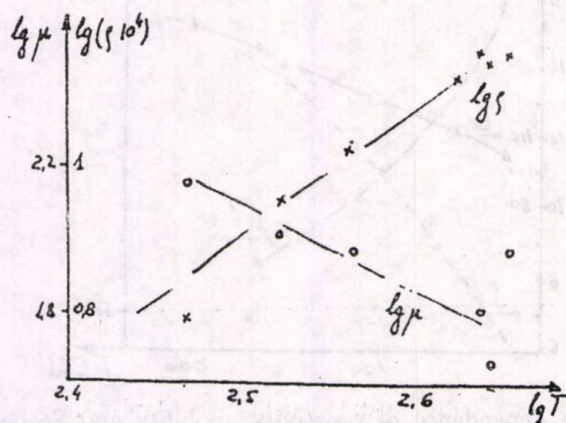


Fig. 3 — The logarithmic dependence between resistivity, mobility and temperature, for the same bulk sample.

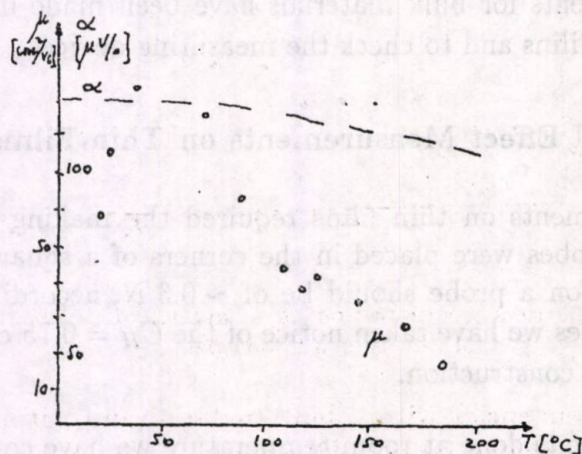


Fig. 4 — The dependence of mobility and Seebeck coefficient as a function of temperature for the ${}^3\text{K}_P^{14}$ thin film.

If we continue to analyze the dependence of mobility and resistivity as a function of temperature, we are able to find out information about the spreading process within the layer.

The experimental data presented in Fig. 7 lead to a relationship such as $\mu \sim T^{-3.3}$, instead of $\mu \sim T^{-2.05}$ within the bulk material. The increase in the exponent could not be caused only by spreading on dislocations for which $\mu \sim T^{-1/2}$, but may be induced by the spreading on the sides of the film and by the possible space intergranular charge.

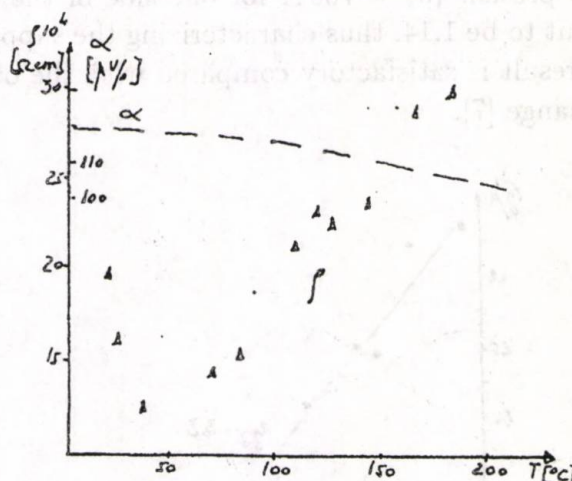


Fig. 5 — The temperature dependence of resistivity and Seebeck coefficient for the thin film sample.

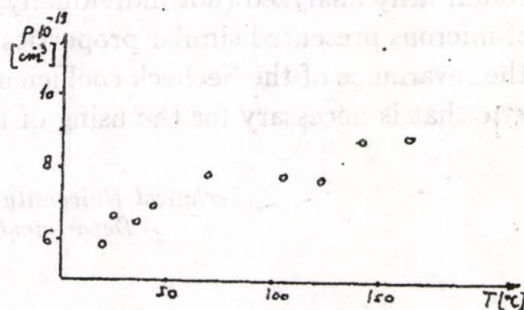


Fig. 6 — Carriers' concentration as a function of temperature for the thin film sample.

From using the dependencies within bulk material and within the film in the Vijay Kumar calculating method [6] we get

$$(7) \quad \alpha = \alpha_m - (B - Cd)^{-1},$$

where

$$B = \frac{1 - 1/\beta}{(A_{ms} + \phi_s)k_B/e}, \quad \beta = \frac{\sigma_s}{\sigma_m},$$

(8)

$$\phi_s = \frac{E_{vs} - E_{vm}}{k_B T}, \quad C = \left[2\beta d_s (A_{ms} + \phi) \frac{k_B}{e} \right]^{-1}$$

and $A_{ms} = A_m - A_s$ have values that depend on the spreading type within bulk material and on the surface, d_s is the dimension of the film in which the influence of the space charge is present ($d_s = 760 \text{ \AA}$ for one side of the materials on Bi_2Te_3 basis). A_{ms} is found out to be 1.14, thus characterizing the supplementary spreading within the film. The result is satisfactory compared with the one previously found, 1.25, from mobility change [7].

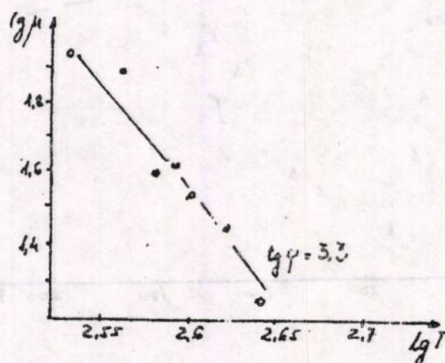


Fig. 7 — Logarithmic dependence between mobility and temperature.

The analyzed sample can be considered as a thick layer, so it belongs to the category of those which are unitarily analyzed (not individually, as those very thin). All film with a thickness of microns presented similar properties. The importance of these films is stressed by the invariance of the Seebeck coefficient with regard to the temperature, a characteristic that is necessary for the using of these films.

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REFERENCES

1. Jeflea A., Teză de doctorat, Universitatea "Al. I. Cuza" Iași, 1990.
2. Catalano S. S., I.E.E.E. Trans. V.E.D., 2 (1963)
3. Goltzmann B. M., Kudinov V. A., Smirnov I. A., *Poluprovodnikovie termoelektricheskie materialy po osnove Bi_2Te_3* . Nauka, Moskva, 1973.
4. Petritz R. L., Phys. Rev., 79, 1023 (1950).
5. Anderson I. C., Thin Solid Film, 18, 239 - 245 (1973).
6. Vijay Kumar A., Chandhuri A. K., Czech. J. Phys. B., 31 (1981).
7. Jeflea A., Sesiunea internă a Institutului Politehnic din Iași, 1991.

MOBILITATEA PURTĂTORILOR DE SARCINĂ ÎN STRATURI
SUBȚIRI DE TIP $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$

(Rezumat)

Realizarea straturilor subțiri termoelectrice competitive sub aspectul proprietăților termoelectrice, cu material masiv, a necesitat urmărirea diferitelor proprietăți specifice și raportarea lor la cele ale materialului masiv. Unul din parametri urmăriți, cel prezentat în lucrare, este mobilitatea purtătorilor de sarcină și dependența sa de temperatură. Materialul termoelectric analizat este $\text{Bi}_{0.5}\text{Sb}_{1.5}\text{Te}_3$. Straturile subțiri analizate au grosimi de $7,452\ \mu\text{m}$ și au fost obținute printr-un procedeu combinat de pulverizare catodică și evaporare flash.

PHOTOVOLTAIC PROPERTIES OF Cu_2S – CdS CELLS

BY

SOFIA MELINTE, ALEXANDRINA JEFLEA and D. MELINTE

Abstract. This study presents an analysis of the various parameters that determine the conversion efficiency of the CdS photovoltaic cells, especially CdS – Cu_2S . Considering the cells previously made, we submit that the fundamental parameters determinants of the efficiency of the solar cells (I_{sc} , V_{OC} , FF) depend on secondary parameters like the recombining speed to interfaces and to volume, the absorption coefficient, the efficiency of the electric load carriers collect, the relaxation and transmission coefficient of the contact grille, the length of the diffusion path of the electric load carriers, etc.

Key words: thin films, fotovoltaic effect, direct conversion.

1. Introduction

From the study and the production of solar cells based on photovoltaic conversion and used as electric energy source it resulted that the parameters determining their efficiency are: the short-circuit current, I_{sc} ; the open-circuit voltage V_{OC} ; the filling or form factor FF. From the works presented in the literature [1]–[9] on the Cu_2S – CdS cells, it results that these parameters depend on a lot of factors, each of which can be controlled for increasing the efficiency. This work is meant to present the values of the three parameters determining the cell efficiency and to render evidently the most important factors effecting these variables.

2. Experimental Results

The Cu_2S – CdS cells already studied were prepared according to the technology described in [1], and consisting in using a metallic substrate above which CdS of high purity is evaporated; then, by a chemical reaction the Cu_xS layer is realized. In order to obtain a small contact resistance, the second electrode was grid shaped. After an adequate thermal treatment different cells had been studied, their characteristic resistance and variation with the illumination being determined, along with the spectral response [5] and the current-voltage characteristics [4].

From the studies undertaken by us the parameters determining the cell efficiency were deduced, namely: the open circuit voltage V_{OC} , that varied within the range 300...540 mV, depending on the irradiation and on the thermal treatment duration. The corresponding short-circuit current, I_{sc} , changed between 15 mA and 35.2 mA, depending on the illumination and the thermal treatment duration. It was established that the short-circuit current increase with duration of the thermal treatment, without to exceed 35.2 mA. The small values of the currents obtained by us are especially due both to the very short dipping time, that did not exceed 4 sec and the short duration of the thermal treatment. At the same time, it has been established that the open-circuit voltage sharp increases with the increase of light intensity then remains constant, while the current increases proportional with the illumination. If the cell temperature increases, a sharp increase of the current and a decrease of the open-circuit voltage are noted.

The form factor FF for the photovoltaic cells that were studied [3]–[5], [8] changed between 0.3 and 0.63 and its increase with the increase duration of the thermal treatment was noted.

3. The Factors Affecting the Cell Efficiency

It is well known that the maximum cell power is given by the relation

$$(1) \quad P_{\max} = I_{sc} V_{CD} FF,$$

where I_{sc} is the short-circuit current; V_{OC} is the open-circuit voltage and FF the form of filling factor of the photovoltaic cell. Even if these parameters are not independent on each other, each of them depends on a series of particular factors [1] over which may be to act in order to obtain a higher efficiency.

3.1. Short-Circuit Current I_{sc}

From the photovoltaic response of the photocells [6], on the basis of energy band diagram given in Fig. 1, it follows that by the absorption of the photons of an energy $h\nu > E_{g1}(\text{Cu}_2\text{S})$ and by electron scattering on $\text{Cu}_2\text{S} - \text{CdS}$ interface, a photovoltaic current will be generated. Part of the carriers entering CdS are oriented by the internal field created within the space charge region and parts of them recombine with holes at $\text{Cu}_2\text{S} - \text{CdS}$ interface.

Therefore, the short-circuit current is almost completely generated within the Cu_2S layer. In the relation (2), which give the short-circuit current,

$$(2) \quad I_{sc} = j_{sc} S_{\perp};$$

$S_{\perp}$ is the measured area of the photovoltaic cell, and the short-circuit current density, j_{sc} is given by the relation

$$(3) \quad j_{sc} = \frac{\mu_2 E_2(\phi)}{v_r + \mu_2 E_2(\phi)} \int_{\lambda=0}^{\lambda_g} \phi_0(\lambda) T_g [1 - R(\lambda)] [1 - A(\lambda)] \eta_{col} [\alpha(\lambda), L, d_1, E_1, r, v] d\lambda,$$

where μ_2 is the carrier mobility in CdS; E_2 is the junction electric field which depends on the total photonic flux ϕ falling on the CdS – Cu₂S interface; v_r is the recombination rate at the CdS – Cu₂S interface; λ_g represents the wavelength corresponding to the gap region for Cu₂S ($\lambda_g = hc/E_{g1}$); $\phi_0(\lambda)$ is the density of the incident photonic flux falling on the cell; T_g is the transmission coefficient of the contact grid; $R(\lambda)$ is the reflection coefficient of the cell; $A(\lambda)$ is the absorption loss coefficient of the photocell; η_{col} is the collection efficiency of the Cu₂S layer, which depends, as seen from relation (3), on: $\alpha(\lambda)$ – i.e. the absorption factor; L – the scattering length; d_1 – the layer thickness; r – the grain size, the drift field E_1 and the superficial recombination rate on the input face of the Cu₂S [5], [7] and which depends from the way of illuminating of the cell (through the copper sulphide or cadmium sulphide). In the relation (3) it results that j_{sc} is variable depending on a diversity of factors over which may be to act.

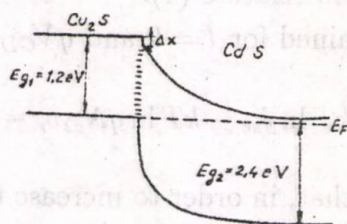


Fig. 1 — The band gap of Cu₂S–CdS solar cell.

The first factor in relation (3) is the interface collection coefficient [7]–[9] and determines what amount of the electrons arrived at the junction is retained in CdS and what amount of electrons is going back to Cu₂S, generating the interface states. A number of parameters are difficult to be determined and others can not be known very precisely, for example $\alpha(\lambda)$ and L , which depend on the stoichiometry of the Cu₂S layer (which in fact is Cu_xS and gives good results only for $x > 1.99$). In our case, we think that this condition was not satisfied, which affected considerably the short-circuit current $L = 0.275 \mu\text{m}$ for CdS layer, when the incident light enters through the transparent metal on CdS (in our case an aluminum layer 12.5 nm thick).

The losses due to the superficial recombination can be eliminated by generating a Cu₂O layer on the Cu₂S face, but the volume losses are quite important, going up to 15% [5]–[7]. For a thickness of Cu₂O layer ($d \approx 30 \text{ \AA}$) $j_{sc} = 18 \text{ mA/cm}^2$, $\text{FF} = 0.63$, $V_{OC} = 0.54 \text{ V}$.

Another loss factor is given by the absorption of the photons within CdS layer, especially of those having the energy lower than the energy of the forbidden region.

It follows from above that, acting on some factor by means of the thermal treatment, the corresponding stoichiometric concentration and the use of very pure materials with the adequate technology, a series of controllable losses can be reduced.

3.2. Open Circuit Voltage V_{OC}

This parameter determines directly the cell power, therefore its efficiency can be theoretically determined from the expression of the current through the cell [6]

$$(4) \quad I = qS_j N_{c2} v_r e^{\frac{E_{g1} - \Delta\chi}{kT}} \left(e^{\frac{q(V - IR_s)}{kT}} - 1 \right) - S_{\perp} j_{sc},$$

where q is the electronic charge S_j is the area of the junction (that may be much greater than $S_{\perp}$ due to the generation of Cu_2S layer) $N_{c2} \approx 2 \cdot 10^{18} \text{ cm}^{-3}$ is the effective density of the state at the limit of the forbidden region of CdS ; v_r is the interface recombination rate that depends on the structural differences between the lattice constants of Cu_2S in CdS , noted by Δa and what, for $\Delta a/a = 0.04$ [6], was found $v_r \approx 10^6 \text{ cm/s}$; $E_{g1} = 1.2 \text{ eV}$ (Cu_2S); $\Delta\chi \approx 0.2 \text{ eV}$ is the difference of the electron affinity of Cu_2S and CdS ; V is the applied voltage or the voltage drop on the load; R_s is the series resistance of the cell. It was supposed that the shunt resistance R_{sh} was very high (junction with $R_{sh} \gg 10^3 \Omega$ can be easily obtained), so that $(V - IR_s)/R_{sh}$ was neglected in relation (4).

The open circuit voltage is obtained for $I = 0$ and $qV_{CD} \gg kT$,

$$(5) \quad qV_{OC} = E_{g1} - \Delta\chi + kT \ln j_{sc} - kT \ln qN_{c2}v_r - kT \ln \frac{S_j}{S_{\perp}}.$$

From this relation one can see that, in order to increase the open circuit voltage, one can act, in fact, on the following parameters: $\Delta\chi$, v_r and S_j . Since $\Delta\chi$ and v_r depend on utilized material, they can be only modified by changing the material, i.e. by using materials with $\Delta\chi$ and v_r as small as possible.

Another factor on which one can act to modify the voltage V_{OC} is the effective junction area S_j . By using another deposition technology for the Cu_2S layer which should lead to the decrease of this area, one can reach a value of the open circuit voltage of $V_{OC} \approx 0.54 \text{ V}$.

However, another factor that affects the open circuit voltage is the existence of a drift field within the Cu_2S which produces diffusion voltage within this layer, leading to a decrease of the open circuit voltage V_{OC} .

3.3. Field Factor FF

The filling or form factor FF is defined as the point of maximum power

$$(6) \quad \text{FF} = \frac{V_m I_m}{V_{OC} I_{sc}}.$$

The relation by which this factor is exactly defined is quite complex [1]. The factors affecting this parameter are: the series resistance R_s depending on the side

leakage of current within Cu_2S layer, as well as on the resistance of CdS layer. An essential contribution to the values of the resistance R_s is brought by the grid contact resistance, as well as by the grid transparency T_g . This transparency is a function on the grid interlining, namely for a grid with parallel lines, $T_g = S/(S + \delta)$, where δ is the grid width and S is the interspace between the lines. It follows that for $\delta/S \leq 1$, a maximum transparency is obtained.

The filling factor increases with the increasing duration of the thermal treatment, what we also found [4].

Concerning the relation (3), it was mentioned that the factor in front of the integral depends on the open circuit voltage and determines the dependence of the current density j_{sc} on this voltage. While it is expected that it should not affect too much the value of the open circuit voltage, since the dependence is logarithmic, it is seriously affecting the filling factor, since the maximum current for the maximum power depends directly on the short-circuit density of current j_{sc} and implicitly on the junction field E_2 .

The field for the case of an illuminated junction [1], [8] can be decomposed into two components; the first one, independent on the voltage, but depending on the wavelength and the bright intensity, and the second, which depends on the voltage. Essentially, the compensation centers from CdS just near the junction are ionized, producing an increase of the positive charge density near the junction, and therefore a decrease of the space charge thickness, leading to an increase of the capacitance of the illuminated cell [8]. The field within the junction exceeds the dark field. This component of the field, resulted from the ionization of the compensation centers does not depend on the voltage while the field generated by the remanent space charge depends on the voltage. All these depend on the duration of the thermal treatment, which finally results in the increase of the filling factor.

All the parameters discussed above influence, directly or indirectly, the cell efficiency, given by the relation:

$$(7) \quad \eta = \frac{j_{sc} V_{OC} FF}{P_{in}} = \frac{I_{sc} V_{OC} FF}{P_{in} A},$$

where $P_{in} \approx 100 \text{ mW/cm}^2$ for the terrestrial conditions (AM1), $j_{sc} = 18 \text{ mA/cm}^2$, $V_{OC} = 0.54 \text{ V}$, $FF = 0.63$; for $A = 1.95 \text{ cm}^2$ (area of cell), results $\eta = 6.12 \%$.

4. Conclusions

On the basis of the results obtained up to now, mentioned in the literature or in our works, dealing with the $\text{CdS} - \text{Cu}_2\text{S}$ cells, we tried to concisely outline in this paper the sum of the factors affecting the efficiency of these cells and the way they can be acted in order to obtain greatly efficient cells.

REFERENCES

1. Rothwarf A., Barnett A. M., IEEE Trans. on Electron Devices, ED-24, 4 (1977).
2. Böer K. W., Phys. Rev., B 13, 5373 (1976).
3. Melinte S., Jeflea Al., *CdS/Cu₂S-Thin Film Solar Cells*. Bul. Inst. Polit. Iași, XXVII (XXXI), 3-4, s. I, 119-123 (1981).
4. Melinte S., Jeflea Al., *The Thin Film MIS Solar Cells Based on CdS*. Proceedings of 4-th Miami International Conference on *Alternative Energy Sources* (T. Veziroglu, Ed.), Vol. 3, *Solar Power Applications Alcohols*, pp. 117-127, 1982.
5. Melinte S., Jeflea Al., Mateescu N., *The Diffusion Length of Minority Carriers in CdS Films Used for Solar Cells*. J. of the Less-Common Metals, 95, 99-103 (1983).
6. Rothwarf A., International Workshop on *Cadmium Sulfide Solar Cells and Other Abrupt Heterojunctions*, University of Delaware, 1975, NSF/RANN/AER 75-15858, p.9.
7. Rothwarf A. et al., Conference Record Eleventh IEEE, *Photovoltaic Specialists Conference*, p. 476, 1975.
8. Melinte S., Jeflea A., Moise M., Mateescu N., *The Study of the Instability of CdS/Cu₂S Solar Cells*. Proceedings of the International Symposium-Workshop on *Renewable Energy Sources-Part A* (T. N. Veziroglu, Ed.), Elsevier Science Publishers, Amsterdam - The Netherlands, 1984, pp. 269-275.
9. Melinte S., Jeflea A., Moise M., Mateescu N., Melinte D., *Alternative Energy Sources*, VI. Vol. 2, *Solar Applications/ Waste Energy*, Springer Verlag, 1985, pp. 297-307.

PROPRIETĂȚI FOTOVOLTAICE ALE CELULELOR DE Cu₂S - CdS

(Rezumat)

Se prezintă o analiză a diversilor parametri ce determină eficiența de conversie a celulelor fotovoltaice pe bază de sulfură de cadmiu, în special CdS - Cu₂S. Pe baza celulelor realizate se arată că parametrii fundamentali ce determină eficiența celulelor solare, cum sunt I_{sc} , V_{OC} , FF, depind de alți parametri secundari cum sunt: viteza de recombinare la interfețe și în volum, coeficientul de absorbție, eficiența colectării purtătorilor de sarcină, coeficientul de relaxare și transmisie al grilei de contact, lungimea drumului de difuzie a purtătorilor de sarcină, etc. Caracterizarea acestor factori, ce determină eficiența de conversie a celulei, constituie interesul lucrării.

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Secția I

MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

RECENZII

BOOK REVIEWS

* * * **Life in the Universe.** Scientific American – A special Issue, W. H. Freeman and Company, New York – Oxford, 1995, 122 pp., \$ 16.95, ISBN 0-7167-2714-5.

The beauty and wonder of the night sky have been made even more awesome as speculation about the origin and evolution of the Universe and has been transformed into solid scientific knowledge over the last century. Scientists have made tremendous strides in answering some of the most compelling questions about the Universe and our place in it, but many remain. How did stars, planets and galaxies form? How did life on Earth begin? Are there other habitats suitable for life in the Universe?

This volume brings together some of the greatest scientists of our time to provide a lucid, accessible overview of the theories and experiments devised to explain the great mysteries of science. Authors such as Steven Weinberg, Carl Sagan, Stephen Jay Gould and Marvin Minsky explore the most intriguing issues: As the Universe continues to expand, what will be its fate? Can we learn enough about our biological, physical and social reality to fashion a future that our planet can sustain? Will robots inherit the Earth? These and many other critical questions are addressed by the very scientists devoted to enhancing our understanding of the laws of nature. They may not be able to answer all of them, but their fascinating explorations take the reader to the very boundaries of scientific knowledge.

C o n t e n t s: *Life in the Universe* (Steven Weinberg); *The Evolution of the Universe* (P. James, E. Peebles, David N. Schramm, Edwin L. Turner and Richard G. Kron); *The Earth's Elements* (Robert P. Kirshner); *The Evolution of the Earth* (Claude J. Allègre and Stephen H. Schneider); *The Origin of Life on the Earth* (Leslie E. Orgel); *The Evolution of Life on the Earth* (Stephen Jay Gould); *The Search for Extraterrestrial Life* (Carl Sagan); *The Emergence of Intelligence* (William H. Calvin); *Will Robots Inherit the Earth?* (Marvin Minsky); *Sustaining Life on the Earth* (Robert W. Kates); *Descartes's Error and the Future of Human Life* (Antonio R. Damasio).

BARRY CIPRA, What's Happening in the Mathematical Sciences. American Mathematical Society, Providence, USA, 1995, 120 pp, \$ 12, ISBN 0-8218-0355-7.

Beautifully produced and marvelously written, **What's Happening in the Mathematical Sciences**, Volume 3, contains 10 articles on recent developments in the field. In an engaging, reader-friendly style, Barry Cipra explores topics ranging from Fermat's Last Theorem to Computational Fluid Dynamics.

The volumes in this series are intended to highlight the many roles mathematics plays in the modern world. Features articles on: the new mathematical method that's taking Wall Street by storm; supercomputing with DNA; how a mathematician studying prime numbers found the famous flaw in the Pentium chip.

C o n t e n s: *Fermat's Theorem – at Last!; A Tale of two Theories; Computer Science Discovers DNA; Divide and Conquer; Mathematics and the Gentle Art of Control; Computational Fluid Dynamics – Verging on Turbulence; Cellular Automata Offer New Outlook in Life, the Universe and Everything; Are Group Theorists Simpleminded?; The Secret Life of Large Numbers; In Math we Trust.*

Unique in kind, lively in style, Volume 3 of **What's Happening in the Mathematical Sciences** is a delight to read and a valuable source of information.

MARVIN JAY GREENBERG, **Euclidean and Non-Euclidean Geometries . Development and History** (Third Edition). W. H. Freeman and Company, New York, 1994, 483 pp., \$ 29.95, ISBN 0-7167-2446-4.

This book presents the *Discovery of Non-Euclidean Geometry and the Subsequent Reformulation of the Foundations of Euclidean Geometry* as a suspense story. The mystery of why Euclid's parallel postulate could not be proved remained unsolved for over two thousand years, until the discovery of non-Euclidean geometry and its Euclidean models revealed the impossibility of any such proof. This discovery shattered the traditional conception of geometry as the true description of physical space.

C o n t e n t s: *Preface; Introduction; 1. Euclid's Geometry; 2. Logic and Incidence Geometry; 3. Hilbert's Axioms; 4. Neutral Geometry; 5. History of the Parallel Postulate; 6. The Discovery of Non-Euclidean Geometry; 7. Independence of the Parallel Postulate; 8. Philosophical Implications; 9. Geometric Transformations; 10. Further Results in Hyperbolic Geometry; Appendix A; Appendix B; Suggested Further Reading; Bibliography; List of Axioms; List of Symbols; Name Index; Subject Index.*

This text is useful for several kinds of students. Prospective high school and college geometry teachers are presented with a rigorous treatment of the foundations of Euclidean geometry and an introduction to hyperbolic geometry (with emphasis on its Euclidean models). General education and liberal arts students are introduced to the history and philosophical implications of the discovery of non-Euclidean geometry (for example, the book was used very successfully as part of a course on scientific revolutions at Colgate University). Mathematics majors are given, in addition, detailed instruction in transformation geometry and hyperbolic trigonometry, challenging exercises, and a historical perspective that, sadly, is lacking in most mathematical texts.

Thus this book is a resource for a wide variety of students, from the naive to the sophisticated, from the nonmathematical-but- educated to the mathematical wizards.

THOMAS F. BANCHOFF, **Beyond the Third Dimension. Geometry Computer Graphics and Higher Dimensions**. W. H. Freeman at Macmillan Press, Oxford – New York, 1996, 211 pp., \$ 19.95, ISBN 0-7167-6015-0.

In our world of three-dimensional space, how can we even attempt to visualize an object of four, five, or six dimensions? Thomas F. B a n c h o f f shows us how with his insightful text, renowned computer graphics and superb illustrations that help us visualize dimensions higher than our own.

Thomas F. B a n c h o f f is Professor of Mathematics at Brown University, where he specializes in the geometry of higher dimensions. In collaboration with his colleagues at Brown's Computer Graphics Laboratory, he has created such award-winning computer-generated films as *The Hypercube: Projections and Slicing* (with Charles S t r a u s s). Prof. B a n c h o f f is the recipient of the 1996 Mathematics Association of America National Award for Distinguished College or University Teaching of Mathematics.

Adding new dimensions to our world, **Beyond the Third Dimension** expands our visual boundaries as we learn how a dimension can represent energy, temperature and numerous other variables, including time. We will see how the concept of dimensions plays a significant, perhaps unexpected role in fields as diverse as medicine and modern art, and also in our everyday lives.

C o n t e n t s: *Preface; 1. Introduction Dimensions; 2. Scaling and Measurement; 3. Slicing and Contours; 4. Shadows and Structures ; 5. Regular Polytopes and Fold-outs; 6. Perspective and Animation; 7. Configuration Spaces; 8. Coordinate Geometry; 9. Non-Euclidean Geometry and Nonorientable Surfaces; Further Readings; Acknowledgments; Illustration Acknowledgments; Sources of Illustrations; Index.*

"This beautiful book gently introduces the reader to the idea of many dimensions" (H. S. M. Coxeter, American Mathematical Monthly).

"Thomas Banchoff has succeeded in producing one of the best popular books on geometry that I have seen. ... all readers, whether lay or professional mathematician, can expect to learn something new in this book" (Nature).

DAVID F. GRIFFITHS and DESMOND J. HIGHAM, **Learning LATEX**. SIAM (Science and Industry Advance with Mathematics), Philadelphia, USA, 1996, 88 pp., \$ 15.50, ISBN 0-89871-383-8.

Here is a short, well-written book that *Covers the Material Essential for Learning LaTeX Without any Unnecessary Detail*. It includes *Incisive Examples* that teach LaTeX in a powerful yet abbreviated fashion. This is the handbook to have if you don't want to wade through extraneous material. This manual includes the following crucial features: *numerous examples* of widely used mathematical expressions; complete documents illustrating the *creation of articles, reports* and overhead projector slides; *troubleshooting tips* to help you pinpoint an error; details of *how to set up a bibliography* and an index; information about *LaTeX resources available on the Internet*.

LaTeX has become an extremely popular type setting system and is widely used throughout the sciences. As a researcher you may need to typeset reports and theses in LaTeX (particularly in any mathematics or computer science discipline). Or you may be someone who had planned to "eventually" get around to learning LaTeX, but you are still using older systems and methods of typesetting. Procrastinate no more!

C o n t e n t s: *Preface; Ch. 1: Preamble. Should You Be Reading this Book?; Motivation; Running LaTeX ; Resources. Ch. 2: Basic LaTeX. Sample Document and Key Concepts; Type Style; Environments; Vertical and Horizontal Spacing. Ch. 3: Typesetting Mathematics; Examples; Equation Environments; Fonts, Hats and Underlining; Braces; Arrays and Matrices; Customized Commands; Theorem-like Environments; Math Miscellany. Ch. 4: Further Essential LaTeX; Document Classes and the Overall Structure; Titles for Documents; Sectioning Commands; Miscellaneous Extras; Troubleshooting; Ch. 5: More about LaTeX; Packages; Inputting Files; Inputting Pictures; Making a Bibliography; Making an Index; Great Moments in LaTeX History. Appendix A: Old LaTeX versus LaTeX 2. Appendix B: A Sample Article. Appendix C: A Sample Report. Appendix D: Slides. Appendix E: Internet Resources. Documentation; CTAN; WWW; Professional Societies; TUG.*

The authors have elected to cover LaTeX 2 the latest standard version at the time of publication. The old and new versions are very similar and it is clear that the LaTeX 2 version will soon dominate. An Appendix discusses the difference between 2 and the older version 2.09.

G. W STEWART, **Afternotes on Numerical Analysis**. SIAM (Society for Industrial and Applied Mathematics), Philadelphia, USA, 1996, 200 pp., ISBN 0-89871-362-5.

There are many textbooks to choose from when teaching an introductory numerical analysis course, but there is only one **Afternotes on Numerical Analysis**. This book presents the *Central Ideas of Modern Numerical Analysis in a Vivid and Straightforward Fashion with a Minimum of Fuss and Formality*. Stewart designed this volume while teaching an upper-division course in introductory numerical analysis. To clarify what he was teaching, he wrote down each lecture immediately after it was given. The result reflects the wit, insight and verbal craftsmanship which are hallmarks of the author.

Simple examples are used to introduce each topic, then the author quickly moves on to the discussion of important methods and techniques. With its rich mixture of graphs and code segments,

the book provides insights and advice that help the reader avoid the many pitfalls in numerical computation that can easily trap an unwary beginner.

Written by a leading expert in numerical analysis, this book is certain to be the one you need to guide you through your favorite textbook.

This book may be used as a supplementary text for upper-division undergraduate and beginning graduate introductory numerical analysis courses.

URI M. ASCHER, **Numerical Solution of Boundary Value Problems for Ordinary Differential Equations**. SIAM (Society for Industrial and Applied Mathematics), Philadelphia, USA, 1995, 595 pp., ISBN 0-89871-354-4.

Our knowledge and understanding of methods for the numerical solution of boundary value problems (BVPs) for ordinary differential equations has increased significantly in the past few years. Although important theoretical and practical developments have taken place on a number of fronts, they have not previously been comprehensively described in any text. This book was written with the intention of bringing together *Basic Theory and Practice and More Advanced Recent Developments* concerning these methods.

This field has been rather late in maturing as compared with the numerical solution of initial value problems (IVPs). In fact, years ago many considered BVPs as some sort of a subclass of IVPs, wherein one fiddles with the initial conditions in order to get things right at the other end. It gradually became clear that IVPs are actually a special and in some sense relatively simple subclass of BVPs. The fundamental difference is that for IVPs one has complete information about the solution at one point (the initial point), so one may consider using a marching algorithm which is always local in nature. For BVPs, on the other hand, no complete information is available at any point, so the end points have to be connected by the solution algorithm in a global way. Only after stepping through the entire domain can the solution at any point be determined.

The first three chapters of the book are preliminary to the actual introduction of the numerical methods. The preliminaries often involve some rather nontrivial mathematics.

Ch. 1 consists of two parts. In the first some *Basic Forms for BVPs* which are treated throughout the book are *Introduced*. The second gives a *Representative Collection of BVPs*. A *Background in Ordinary Differential Equations* (ODEs), *Linear Algebra and Basic Numerical Analysis* is gathered in Ch. 2. Ch. 3 is devoted to *Theory of ODEs* with the purpose to introduce the material necessary for the development and understanding of numerical methods presented later on.

In Ch. 4 the *First Class of Numerical Methods* for the solution of BVPs are described, based on solving related initial value problems (IVPs) in order to obtain a BVP solution. The simplest is single shooting, followed by other methods, which moderate the numerical instability with various degrees of success. Ch. 5 describes the *Other Class of Basic Numerical Methods* used in the book, developing essentially the methods from scratch (using Taylor expansions).

Ch. 6 is devoted to the *Fundamental Principle of Decoupling*, treating more advanced theoretical considerations than the previous two chapters. Ch. 7 presents *Numerical Methods for Solving the Linear Systems of Equations Resulting from the Discretizations* (of linear or linearized BVPs) described in Chs. 4 and 5.

The topic of Ch. 8 is the *Numerical Solution of the Nonlinear Equations* resulting from discretizing nonlinear BVPs. The basic method considered in this book is Newton's method. *Mesh Selection* is the topic of Ch. 9. In this chapter, and to a lesser extent in the previous one, the presentation is more practically oriented and less theoretical.

The current research on *Numerical Methods* for IVPs focuses on *Stiff Problems*, since these are considered to be the ones not yet handled satisfactorily in practice. The corresponding research in BVPs has claimed front stage much more recently, but with much vigor, and this is evidenced by the size of Ch. 10, devoted to this subject. A number of *Related Topics*, several being important enough to warrant a book in their own right, are briefly described in Ch. 11.

This book was written for a broad range of readers, including the senior undergraduate or graduate student, the scientist or engineer with a practical orientation and probably a specific

application in mind, the purer analyst whose interest lies in the mathematical properties of the various numerical methods, and the researcher working at the frontiers in this subject.

STEPHEN BOYD, LAURENT EL GHAOU, ERIC FERON and VENKATARAMANAN BALAKRISHNAN, **Linear Matrix Inequalities in System and Control Theory**. SIAM (Society for Industrial and Applied Mathematics), Philadelphia, USA, 1994, 193 pp., ISBN 0-89871-334-X.

The basic topics of this book is *Solving Problems from System and Control Theory Using Convex Optimization*. A wide variety of problems arising in system and control theory can be reduced to a handful of standard convex and quasiconvex optimization problems that involve matrix inequalities. For a few special cases there are "analytic solutions" to these problems, but the authors main point is that they can be solved numerically in all cases. These standard problems can be solved in polynomialtime (by, e.g., the ellipsoid algorithm of Shor, Nemirovskii and Yudin), and so are tractable, at least in a theoretical sense. Recently developed *Interior-Point Methods* for these standard problems have been found to be *Extremely Efficient in Practice*.

This book is primarily intended for the researcher in system and control theory, but can also serve as a source of application problem for researchers in convex optimization. The methods described in this book have great practical value. This is a research monograph, presenting no specific examples or numerical results, but only brief comments about the implications of the results for practical control engineering. To put it in a more positive light, this book will later be considered as the first book on the topic, not the most readable or accessible.

RITA G. LERNER and GEORGE L. TRIGG, **Encyclopedia of Physics**. VCH Publishers, Inc., 1990, 1048 pp., ISBN 0-89573-752-3.

Like the first edition, this **Encyclopedia** describes physics only as of a particular moment in time. In the ten years since that edition appeared there have been significant changes in many areas of physics.

The most publicized has been the discovery of *Materials that Become Superconductors of Electricity* at temperatures much higher than previously known. But other developments have been of at least equal importance within physics, if not so striking to the rest of the world. *Fractal Geometry* has been developed to describe irregular shapes and forms, and has been applied to describing physical phenomena; their beauty is apperant to all. *Chaotic Theory*, first explored early in this century, is being used to describe dynamical systems, especially random ones. The study of *Critical Phenomena* has been enlarged by the inclusion of dynamic aspects.

The understanding of the *Solid State* has extended to structures with such names as intermediate-valence compounds and heavy-fermion materials. *Scanning Tunneling Microscopy* has led to new insights in surface physics. An established effect, the *Hall Effect* has been found to have intriguing *Quantum Aspect* which have increased in potential for application to measurement standards.

The *Appearance of Supernova 1987a* has excited astronomers and stimulated new theoretical work in astronomy and cosmology. Theoretical physicists have continued their attempts to derive a *Grand Unified Theory* linking the strong, weak, and electromagnetic interactions and perhaps, through a so-called *Supersymmetry*, to link them to gravitation. And arrangements known as *Cellular Automata* have moved computers a small step closer to mimicking some of the brain's operations.

As in the previous edition, articles are arranged in alphabetical order, with some titles inverted in order to group related articles near each other. At the end of most articles, cross references are given to related articles in this volume. Since it is not possible in a volume of this size to explore any one subject in depth, *most of the authors have provided references for further reading*; these are usually labeled E, I, or A indicating that the reference is elementary, intermediate, or advanced.

DAVID PEAK and MICHAEL FRAME, **Chaos Under Control. The Art and Science of Complexity**. W. H. Freeman at Macmillan Press, New York, 1994, 408 pp., \$ 24.95, ISBN 0-7167-2429-4.

The reader may not be fluent in math and physics but only curious about the *Dynamics of Order* and *Chaos and the Geometry of Fractals* and to understand what these ideas reveal about the natural world.

Chaos Under Control introduces to the *Rich Concepts Driving the New Science of Complexity*, allowing to discover the principles behind some of the most talked-about research and findings of our time. Some astonishing examples and applications of these concepts can be seen and the accompanying software (available from the authors) can turn a Macintosh computer into a window of discovery, through which can be watched fractal and chaotic phenomena coming to life visually.

The authors, David Peak and Michael Frame, teach at Union College, have both extensive teaching and research experience and have published numerous research papers being known for their skill in communicating science to the general public.

C o n t e n t s: Foreword by Benoit B. Mandelbrot; Foreword by Rhonda Roland Shearer. Preface; Introduction: A First Word. Ch. 1, Art and Magic. Ch. 2, Only God Can Make a Tree? Ch. 3, Unearthly Earthly Dimensions. Ch. 4, Chaos. Ch. 5, The Order Within. Ch. 6, Hide and Seek. Ch. 7, Benoit's World. Ch. 8, The Fragility of Order. Ch. 9, Some Call it Life. Ch. 10, Certainly Not the Last Word. References and Notes. Explorations and Challenges. Index.

"After finishing *Chaos Under Control*, the nonspecialist, in particular, will come away not as a passive reader but as a knowing participant, painlessly sharing the vision of the technical experts "in the known" at the frontiers of art and science" (Rhonda Roland Shearer).

CHARLES W. MISNER, KIP S. THORNE and JOHN ARCHIBALD WHEELER. **Gravitation**. W. H. Freeman and Company, New York - Oxford, 1995, 1279 pp., \$ 52.95, ISBN 0-7167-0344-0.

Put as simply as possible, this book is a book on *Einstein's Theory of Gravity* (General Relativity). It is the first textbook on the subject that uses throughout the modern formalism and notation of differential geometry, and it is the first book to document in full the revolutionary techniques developed during the past decade to test the theory of general relativity (such tests include the use of atomic clocks, long baseline radio interferometry, interplanetary radar, lunar laser ranging, and interplanetary spacecraft).

Among the topics treated in depth are *Relativistic Stars and Star Clusters* and their possible roles in pulsars and quasars, recent developments in *Cosmology, Gravitational Collapse and Black Holes*, and the *Generation, Propagation and Detection of Gravitational Waves*.

All three authors are uniquely qualified to have written this authoritative work: Charles W. Misner is Professor of Physics at the University of Maryland, Kip S. Thorne is Professor of Theoretical Physics at the California Institute of Technology and Adjunct Professor of Physics at the University of Utah and John Archibald Wheeler is Joseph Henry Professor of Physics at Princeton University. During the past decade, they and their research groups have been responsible for approximately half of the theoretical developments in the areas of concentration mentioned.

C o n t e n t s: Part I, *Spacetime Physics*. Part II, *Physics in Flat Spacetime*. Part III, *The Mathematics of Curved Spacetime*. Part IV, *Einstein's Geometric Theory of Gravity*. Part V, *Relativistic Stars*. Part VI, *The Universe*. Part VII, *Gravitational Collapse and Black Holes*. Part VIII, *Gravitational Waves*. Part IX, *Experimental Tests of General Relativity*. Part X, *Frontiers. Bibliography and Index of Names. Subject Index*.

Here is a book that is without competition and without peer - a book that interested in modern science, bearing in mind also that "Einstein's description of gravitation as curvature of space time led directly to the greatest of all predictions of his theory, that the Universe itself is dynamic. Physics still has far to go to come to terms with this amazing fact and what it means for man and his relation with the Universe" (John Archibald Wheeler).

R. K. PATHRIA, **Statistical Mechanics** (Second Ed.). Butterworth-Heinemann, Linacre House, Jordan Hill, Oxford, England, 1996, 512 pp., ISBN 0-7506-2469-8.

This text is designed for postgraduate courses in Statistical Mechanics, and *Provides a Basic Grounding in a Manner that Brings out the Essence of the Subject with Due Rigour* but without undue pain.

This classic text, which was first published in 1972 and has continued to be popular ever since, has now been brought up to date by incorporating the remarkable *Developments in the Field of Phase Transitions and Critical Phenomena* that have taken place in the intervening years. This has been done by adding three new chapters which will enhance the usefulness of this book both for students and instructors.

Widely acclaimed for its clean derivations and clear explanations, **Statistical Mechanics** in its second edition will continue to provide further generations of students with a solid training in the methods of statistical physics.

Y. S. TANG and JAMES H. SALING, **Radioactive Waste Management**. Hemisphere Publishing Corporation, New York, 1990, 460 pp., ISBN 0-89116-666-1.

During the first *Conference on the Peaceful Uses of Atomic Energy*, held in Geneva in 1955, management concepts were introduced and discussed. Subsequently, several management techniques have been developed and put into practice. Only in the recent past has the *Management of Radiactive* become a matter of great public concern. As the well-known geochemist Konrad B. K r a u s k o p f, Stanford University Professor Emeritus and an advisor to federal agencies concerned with waste disposal, has said in public, nuclear waste disposal could be done safely. The problem is political not technological. This point of view has been shared by many scientists and engineers, including Edward T e l l e r.

The National Research Council, through the Waste Isolation Systems Panel, made a study of the isolation system for geological disposal of radwaste for the U. S. Department of Energy (DOE). Among other overall conclusions, the panel's report indicated that the waste technology identified in the study should be more than adequate for isolating radwastes from the biosphere and for protecting public health and safety. Public acceptance and cooperation are indispensable for the success of this program and the public is apprehensive about the potential hazard of radwaste material. The importance of *Understanding the Management Technologies, Design Philosophy and Performance Evaluation Related to Processing, Packaging, Storing, Transporting and Disposing of Radewaste* cannot overemphasized and such information must be disseminated.

The purposes of the present book are: to create a general awareness of technologies and programs of radioactive waste management; to summarize the current status of such technologies; to prepare practicing scientists, engineers, administrative personnel, and students for the future demand for a working team in such waste management.

C o n t e n t s: 1, *Introduction*. 2, *Radiation Sources, Exposure and Health Effects*. 3, *Spent Fuel Management*. 4, *High-Level Waste Management*. 5, *Transuranic (TRU) Waste*. 6, *Low-Level Waste*. 7, *Uranium Ore Mill Tailings Management*. 8, *Transportation*. 9, *Decontamination and Decommissioning*. 10, *Alternative Waste Management Techniques: Seabed Disposal*.

This book is aimed at serving as textbook for students in the energy field and a reference book for those who play the decision-making roles at different levels of the government.



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